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A

THEORETICAL AND PRACTICAL

TREATISE ON

ALGEBRA:

IN WHICH THE EXCELLENCIES OF THE DEMONSTRATIVE METHODS OF THE
FRENCH ARE COMBINED WITH THE MORE PRACTICAL OPERATIONS
OF THE ENGLISH; AND CONCISE SOLUTIONS POINTED
OUT AND PARTICULARLY INCULCATED.

DESIGNED FOR

SCHOOLS, COLLEGES, AND PRIVATE STUDENTS.

Horatio

BY H. N. ROBINSON, A. M.,

FORMERLY PROFESSOR OF MATHEMATICS IN THE UNITED STATES NAVY; AUTHOR
OF A TREATISE ON ARITHMETIC, ASTRONOMY, GEOMETRY, TRIGONOMETRY,
SURVEYING, CALCULUS, NATURAL PHILOSOPHY, ETC., ETC.

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PREFACE.

Some apology may appear requisite for offering a new book to the public on the science of Algebra—especially as there are several works of acknowledged merit on that subject already before the public, claiming attention.

But the intrinsic merits of a book are not alone sufficient to secure its adoption, and render it generally useful. In addition to merit, it must be adapted to the general standard of scientific instruction given in our higher schools; it must conform in a measure to the taste of the nation, and correspond with the general spirit of the age in which it is brought forth.

The elaborate and diffusive style of the French, as applied to this science, can never be more than *theoretically* popular among the English; and the severe, brief, and practical methods of the English are almost intolerable to the French. Yet both nations can boast of men highly pre-eminent in this science, and the high minded of both nations are ready and willing to acknowledge the merits of the other; but the style and spirit of their respective productions are necessarily very different.

In this country, our authors and teachers have generally adopted one or the other of these schools, and thus have brought among us difference of opinion, drawn from these different standards of measure for true excellence.

Very many of the French methods of treating algebraic science are not to be disregarded or set aside. First principles, theories and demonstrations, are the essence of all true science, and the French are very elaborate in these. Yet no effort of individuals, and no influence of a few institutions of learning, can change the taste of the American people, and make them assimilate to the French, any more than they can make the entire people assume French vivacity, and adopt French manners.

Several works, modified from the French, have had, and now have considerable popularity, but they do not naturally suit American pupils. They are not sufficiently practical to be unquestionably popular; and excellent as they are, they fail to inspire that enthusiastic spirit, which works of a more practical and English character are known to do.

At the other extreme are several English books, almost wholly practical, with little more than arbitrary rules laid down. Such books may in time make good *resolvers* of problems, but they certainly fail in most instances to make scientific algebraists.

The author of this work has had much experience as a teacher of algebra and has used the different varieties of text books, with a view to test their comparative excellencies, and decide if possible on the standard most proper to be adopted, and of course he designed this work to be such as his experience and judgment would approve.

One of the designs of this book is to create in the minds of the pupils a love for the study, which must in some way be secured before success can be attained. Small works designed for children, or those purposely adapted to persons of low capacity, will not secure this end. Those who give tone to public opinion in schools, will look *down upon*, rather than *up to*, works of this kind, and then the day of their usefulness is past. On the other hand, works of a high theoretical character are apt to discourage the pupil before his acquirements enable him to appreciate them, and on this account alone such works are not the most proper for elementary class books.

This work is designed, in the strictest sense, to be both theoretical and practical, and therefore, if the author has accomplished his design, it will be found about midway between the French and English schools.

In this treatise will be found condensed and brief modes of operation, not hitherto much known or generally practised, and several expedients are systematised and taught, by which many otherwise tedious operations are avoided.

Some applications of the celebrated problems of the couriers, and also of the lights, are introduced into this work, as an index to the pupil of the subsequent utility of algebraic science, which may allure him on to more thorough investigations, and more extensive study.

Such problems would be more in place in text books on natural philosophy and astronomy than in an elementary algebra, but the almost entire absence of them in works of that kind, is our apology for inserting them here, if apology be necessary.

Quite young pupils, and such as may not have an adequate knowledge of physics and the general outlines of astronomy, may omit these articles of application; but in all cases the teacher alone can decide what to omit and what to teach.

Within a few years many new text books on algebra have appeared in different parts of the country, which is a sure index that something is desired—something expected,—not yet found. The happy medium between the theoretical and practical mathematics, or, rather, the happy blending of the two, which all seem to desire, is most difficult to attain; hence, many have failed in their efforts to meet the wants of the public.

Metaphysical theories, and speculative science, suit the meridians of France and Germany better than those of the United States. But it is almost impossible to comment on this subject without being misapprehended; the author of this book is a great admirer of the pure theories of algebraical science, for it is impossible to be practically skillful without having high theoretical acquirements. It is the man of theory who brings forth practical results but it is not theory alone—it is theory long and well applied.

Who will contend that Watt, Fitch, or Fulton, were ignorant or inattentive to every theory concerning the nature and power of steam, yet they are only known as practical men, and it is almost in vain to look for any benefactors of mankind, or any promoters of real science from those known only as theorists, or among those who are strenuous contenders for technicalities and forms.

We are led to these remarks to counteract, in some measure, if possible, that false impression existing in some minds, that a high standard work on algebra, must necessarily be very formal in manner and abstrusely theoretical in matter; but in our view these are blemishes rather than excellencies.

The author of this work is a great advocate for brevity, when not purchased at the expense of perspicuity, and this may account for the book appearing very small, considering what it is claimed to contain. For instance, we have only two formulas in arithmetical progression, and some authors have 20. We contend *the two* are sufficient, and when well understood cover the whole theory pertaining to the subject, and in practice, whether for absolute use or lasting improvement of the mind, are far better than 20. The great number only serves to confuse and distract the mind; the two essential ones, can be remembered and most clearly and philosophically comprehended. The same remarks apply to geometrical progression.

In the general theory of equations of the higher degrees this work is not too diffuse; at the same time it designs to be simple and clear, and as much is given as in the judgment of the author would be acceptable, in a work as elementary and condensed as this; and if every position is not rigidly demonstrated, nothing is left in obscurity or doubt.

We have made special effort to present the beautiful theorem of Sturm in such a manner as to bring it direct to the comprehension of the student, and if we have failed in this, we stand not alone.

The subject itself, though not essentially difficult, is abstruse for a learner, and in our effort to render it clear we have been more circuitous and elaborate than we had hoped to have been, or at first intended.

We may apply the same remarks to our treatment of Horner's method of solving the higher equations.

Brevity is a great excellence, but perspicuity is greater, and, as a general thing, the two go hand in hand; and these views have guided us in preparing the whole work; we have felt bound to be clear and show the rationale of every operation, and the foundation of every principle, at whatever cost.

The Indeterminate and Diophantine analysis are not essential in a regular course of mathematics, and it has not been customary to teach them in many institutions; for these reasons we do not insert them in our text book. The teacher or the student, however, will find them in a concise form in a key to this work.

PREFACE TO THE 27TH (ENLARGED) EDITION.

This is a progressive age, and teachers, and schools, and books, must progress, proportionally, or be left in the rear.

To keep up with the spirit of the times, and make this volume more complete and valuable, we have added some twenty-four pages of what we conceive to be very valuable matter.

In the demonstration of the binomial theorem we originally adopted that method, which, to judge from our own personal experience and observation as a teacher of others, was the one most easily comprehended by learners—but unfortunately, that method was not complete; it demonstrated only so far as the actual operation was carried, and the further continuation of the law could only be inferred. This being the case, we have added another demonstration in this edition, which may require a little more mental discipline to clearly comprehend, but which is, in itself, perfect and complete.

We have enlarged the practical part of logarithms, and inserted two small tables, which will enable one to obtain the logarithm of any number whatever; and by the examples and illustrations here given, a student can obtain a very good knowledge of logarithms, and their uses, in a smaller compass than can be found in any other book known to us.

We have also added several new practical artifices in the solution of the higher equations, which we hope will not be overlooked. And to the progressive teachers of the United States the whole is respectfully dedicated, by their obedient servant,

THE AUTHOR

Eldridge, N. Y., September, 1857.

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ELEMENTS OF ALGEBRA.

INTRODUCTION.

DEFINITIONS AND AXIOMS.

Algebra is a general kind of arithmetic, an universal analysis, or science of computation by symbols.

Quantity or magnitude is a general term applied to everything which admits of increase, diminution, and measurement.

The measurement of quantity is accomplished by means of an assumed unit or standard of measure; and the unit must be the same, in kind, as the quantity measured. In measuring length, we apply length, as an inch, a yard, or a mile, &c.; measuring area, we apply area, as a square inch, foot, or acre; in measuring money, a dollar, pound, &c., may be taken for the unit.

Numbers represent the repetition of things, and when no application is made, the number is said to be abstract. Thus 5, 13, 200, &c., are numbers, but $5, 13 yards, 200 acres, are quantities.

In algebraical expressions, some quantities may be known, others unknown; the known quantities are represented by the first or leading letters of the alphabet, a, b, c, d, &c., and the unknown quantities by the final letters, z, y, x, u, &c.

THE SIGNS.

(1) The perpendicular cross, thus $+$, called *plus*, denotes *addition*, or a positive value, state, or condition.

(2) The horizontal dash, thus $-$, called *minus*, denotes *subtraction*, or a negative value, state, or condition.

(3) The diamond cross, thus $\times$, or a point between two quantities, denotes that they are to be multiplied together.

(4) A horizontal line with a point above and below, thus $\div$, denotes division. Also, two quantities, one above another, as numerator and denominator, thus $\frac{a}{b}$, indicates that a is divided by b.

(5) Double horizontal lines, thus $=$, represent equality. Points between terms, thus $a : b :: c : d$, represent proportion, and are read as a is to b so is c to d.

(6) The following sign represents root $\surd$; alone it signifies square root. With small figures attached, thus ${}^3\surd$ ${}^4\surd$ ${}^5\surd$, &c., indicates the *third, fourth, fifth, &c.*, root.

Roots may also be represented by fractions written over a quantity, as $a^{\frac{1}{2}}$ $a^{\frac{1}{3}}$ $a^{\frac{1}{4}}$, &c., which indicate the *square root*, the *third root*, and *fourth root* of a.*

(7) This symbol, $a>b$, signifies that a is greater than b.
This " , $a<b$, signifies that a is less than b.

(8) A *vinculum* or bar ——, or parenthesis ( ) is used to connect several quantities together. Thus $\overline{a+b+c}\times x$, or $(a+b+c)x$, denotes that a plus b plus c is to be multiplied by x. The bar may be placed vertically, thus, which is the same as $(a-d+e)\,y$, or the same as $ay-dy+ey$ without the vinculum.

$$\left.\begin{array}{r} a \\ -d \\ +e \end{array}\right| y$$

(9) Simple quantities consist of a single term, as a, b, ab, $3x$, &c. Compound quantities consist of two or more terms connected by their proper signs, as $a+x$, $3b+2y$, $7ab-3xy+c$, &c A binomial consists of two terms; a trinomial of three; and a polynomial of many, or any number of terms above two.

(10) Quantities multiplied together are called *factors* of the product, and *factors* are called *coefficient* in respect to each other. Thus, in the expressions $3x$, abx; 3 and x are factors of the quantity $3x$, and 3 is the coefficient of x. Also, a, b, and x are factors of abx; a is the coefficient of bx; b is the coefficient of ax; and ab is the coefficient of x.

(11) The *measure* of a quantity is some exact *factor* of that quantity: thus, $5a$ is the measure of $20a$ or $25a$.

---

*The adoption and utility of this last mode of notation, which ought to be exclusively used, will be explained in a subsequent part of this work.

(12) The *root* of any quantity is some *equal* factor of that quantity. The square root is one of *two* equal factors; the cube root one of *three* equal factors, and so on.

(13) The *multiple* of any quantity is some exact number of times that quantity.

Other terms will be defined as we use them.

AXIOMS.

Axioms are self-evident truths, and of course are above demonstration; no explanation can render them more clear. The following are those applicable to algebra, and are the principles on which the truth of all algebraical operations *finally rests:*

Axiom 1. If the same quantity or equal quantities be *added* to equal quantities, their *sums* will be equal.

2. If the same quantity or equal quantities be *subtracted* from equal quantities, the *remainders* will be equal.

3. If equal quantities be *multiplied* into the same, or equal quantities, the *products* will be equal.

4. If equal quantities be *divided* by the same, or by equal quantities, the *quotients* will be equal.

5. If the same quantity be both *added* to and *subtracted from* another, the value of the latter will not be altered.

6. If a quantity be both *multiplied* and *divided* by another, the value of the former will not be altered.

7. Quantities which are respectively equal to any other quantity are equal to each other.

8. Like roots of equal quantities are equal.

9. Like powers of the same or equal quantities are equal.

EXERCISES ON NOTATION.

When definite values are given to the letters employed, we can at once determine the value of their combination in any algebraic expression

Let $a=5 \quad b=20 \quad c=4 \quad d=1$

Then $a+b-c=5+20-4$ or $a+b-c=21$

$$\frac{b}{a}+d=\frac{20}{5}+1 \qquad \text{or } \frac{b}{a}+d=5$$

$$\frac{a}{b}+d=\frac{5}{20}+1 \qquad \text{or } \frac{a}{b}+d=\frac{5}{4}$$

$$ab+ac+d=5\times20+5\times4+1=121$$

$$2a+3b+2c+5d=10+60+8+5=83$$

SECTION I.

ADDITION.

(Art. 1.) Before we can make use of literal or algebraical quantities to aid us in any mathematical investigation, we must not only learn the nature of the quantities expressed, but how to add, subtract, multiply, and divide them, and subsequently learn how to raise them to powers, and extract roots.

The pupil has undoubtedly learned in arithmetic, that quantities representing different things *cannot* be added together; for instance, dollars and yards of cloth cannot be put into one sum; but dollars can be added to dollars, and yards to yards; units can be added to units, tens to tens, &c. So in algebra, a can be added to a, making $2a$; $3a$ can be added to $5a$, making $8a$. As a may represent a dollar, then $3a$ would be 3 dollars, and $5a$ would be 5 dollars, and the sum would be 8 dollars. Again, a may represent any number of dollars as well as one dollar; for example, suppose a to represent 6 dollars, then $3a$ would be 18 dollars, and $5a$ would be 30 dollars, and the whole sum would be 48 dollars. Also, $8a$ is 8 times 6 or 48 dollars; hence any number of a's may be added to any other number of a's by *uniting* their coefficients; but a cannot be added to b, or $4a$ to $3b$, or to any other dissimilar quantity, because it would be adding *unlike things*, but we can write $a+b$ and $3a+3b$, indicating the addition by the sign, making a *compound quantity*, Let the pupil observe that a broad generality, a wide latitude must be given to the term *addition*. In algebra, it rather means uniting, condensing, or reducing terms, and in some cases, the sum may *appear* like *difference*, owing to the difference of signs. Thus, $4a$ added to $-a$ is $3a$; that is, the quantities *united* can make only $3a$, because the minus sign indicates that one a must be taken out. Again, $7b+3b-4b$, when united, can give only $6b$, which is in fact the *sum* of these quantities, as $4b$ has the minus sign, which demands that it should be taken out; hence to add similar quantities we have the following

Rule. *Add the affirmative coefficients into one sum and the negative ones into another, and take their difference with the sign of the greater, to which affix the common literal quantity.*

EXAMPLES FOR PRACTICE.

$$\begin{array}{lrrrrr}
 & 5a & 17x & +5ab & 6a+5b & -7cd+8xy \\
 & 2a & 2x & -6ab & -6a-4b & -2cd+3xy \\
\hline
\text{Sum} & 7a & 19x & -ab & +b & -9cd+11xy
\end{array}$$

$$\begin{array}{llcr}
 & 5a+b & cdy+ax & 4x-6 \\
 & 3a+c & 2cdy-3ax & 2x+10 \\
 & 7a-2b+c & 4cdy+3ax & -3x+7 \\
 & -3a-3b-4c & -7cdy-ax & 6x-12 \\
\hline
\text{Sum} & 12a-4b-2c & 0 \quad 0 & 9x-1
\end{array}$$

N. B. Like quantities, of whatever kind, whether of powers or roots, may be added together the same as more simple or rational quantities.

Thus $3a^2$ and $8a^2$ are $11a^2$, and $7b^3+3b^3=10b^3$. No matter what the terms may be, if they are only alike in kind. Let the reader observe that $2(a+b)+3(a+b)$ must be together $5(a+b)$, that is, 2 times any quantity whatever added to 3 times the same quantity must be 5 times that quantity. Therefore, $4\sqrt{x+y}+3\sqrt{x+y}=7\sqrt{x+y}$, for $\sqrt{x+y}$, which represents the square root of $x+y$, may be considered a single quantity.

(Art. 2.) To find the sum of various quantities we have the following

Rule. *Collect together all those that are alike, by uniting their coefficients, and then write the different sums, one after another, with their proper signs.*

EXAMPLES.

$$
\begin{array}{lrrr}
 & \textbf{1.} & \textbf{2.} & \textbf{3.} \\
 & 3xy & 9x^2y & 6xy-12x^2 \\
 & 2ax & -7x^2y & -4x^2+3xy \\
 & -5xy & +3axy & +4x^2-2xy \\
 & 6ax & -4x^2y & -3xy+4x^2 \\
\hline
\text{Sum} & 8ax-2xy & 3axy-2x^2y & 4xy-8x^2
\end{array}
$$

$$
\begin{array}{lll}
\textbf{4.} & \textbf{5.} & \textbf{6.} \\
4ax-130+3\sqrt{x} & 14ax-2x^2 & 9+10\sqrt{ax}-5y \\
5x^2+3ax+9x^2 & 5ax+3xy & 2x+7\sqrt{xy}+5y \\
7xy-4\sqrt{x}+90 & 8y^2-4ax & 5y+3\sqrt{ax}+4y \\
\sqrt{x}+40-6x^2 & 3x^2+26 & 10-4\sqrt{ax}+4y \\
\hline
7ax+8x^2+7xy & &
\end{array}
$$

7. Add $2xy-2a^2$, $3a^2+xy$, a^2+xy, $4a^2-3xy$, $2xy-2a^2$.
Ans. $4a^2+3xy$.

8. Add $8a^2x^2-3xy$, $5ax-5xy$, $9xy-5ax$, $2a^2x^2+xy$, $5ax-3xy$.
Ans. $10a^2x^2+5ax-xy$.

9. Add $2x^{\frac{1}{2}}-10y$, $3\sqrt{xy}+10x$, $2x^2y+25y$, $12xy-\sqrt{xy}$, $-8y+17\sqrt{xy}$.
Ans. $2x^2y+12xy+10x+2x^{\frac{1}{2}}+19\sqrt{xy}+7y$.

10. Add $2bx-12$, $3x^2-2bx$, $5x^2-3\sqrt{x}$, $3\sqrt{x}+12$, x^2+3.
Ans. $9x^2+3$.

11. Add $10b^2-3bx^2$, $2b^2x^2-b^2$, $10-2bx^2$, b^2x^2-20, $3bx^2+b^2$.
Ans. $10b^2+3b^2x-2bx^2-10$.

12. Add $2a^2-3ax^{\frac{1}{2}}+x^2$, $2ax^{\frac{1}{2}}-13xy+8$, $10a^2-xy-4$.
Ans. $12a^2-ax^{\frac{1}{2}}+x^2-14xy+4$.

13. Add $9bc^3-18ac^2$, $15bc^3+ac$, $9ac^2-24bc^3$, $9ac^2-2$
Ans. $ac-2$

14. Add $3m^2-1$, $6am-2m^2+4$, $7-8am+2m^2$, and $6m^2+2am+1$. *Ans* $9m^2+11$.

15. Add $12a-13ab+16ax$, $8-4m+2y$, $-6a+7ab^2+12y-24$, and $7ab-16ax+4m$.

Ans. $6a-6ab+14y+7ab^2-16$.

16. Add $72ax^4-8ay^3$, $-38ax^4-3ay^4+7ay^3$, $8+12ay^4$, $-6ay^3+12-34ax^4+5ay^3-9ay^4$. *Ans.* $-2ay^3+20$.

Add $a+b$ and $3a-5b$ together.

Add $6x-5b+a+8$ to $-5a-4x+4b-3$.

Add $a+2b-3c-10$ to $3b-4a+5c+10$ and $5b-c$.

Add $3a+b-10$ to $c-d-a$ and $-4c+2a-3b-7$.

Add $3a^2+2b^2-c$ to $2ab-3a^2+bc-b$.

(Art. 3.) When similar quantities have *literal* coefficients, we may add them by putting their coefficients in a vinculum, and writing the term on the outside as a factor.

Thus the sum of ax and bx is $(a+b)x$.

EXAMPLES.

$$\begin{array}{lcc} & \textbf{1.} & \textbf{2.} \\ \text{Add} & ax+by^2 & ay+cx \\ & 2cx+3ay^2 & 3ay+2cx \\ & 4dx+7y^2 & 4y\ +6x \\ \hline \text{Sum} & (a+2c+4d)x+(b+3a+7)y^2 & (4a+4)y+(3c+6)x \end{array}$$

$$\begin{array}{lcc} & \textbf{3.} & \textbf{4.} \\ \text{Add} & 3x+2xy & ax+7y \\ & bx+cxy & 7ax-3y \\ & (a+b)x+2cdxy & -2x\ +4y \\ \hline \text{Sum} & (a+2b+3)x+(2cd+c+2)xy & (8a-2)x+8y \end{array}$$

5. Add $8ax+2(x+a)+3b$, $9ax+6(x+a)-9b$, and $11x+6b-7ax-8(x+a)$. *Ans.* $10ax+11x$.

6. Add $(a+b)\sqrt{x}$ and $(c+2a-b)\sqrt{x}$ together.

7. Add $28a^3(x+5y)+21$, $18a-13a^3(x+5y)$, $-15a^3(x+5y)-8$. *Ans.* $18a+13$

8. Add $17a(x+3ay)+12a^3b^4c^2$, $8-18ay-8a^3b^4c^2$, $-7a(x+3ay)-4+17ay$. *Ans.* $10a(x+3ay)+4a^3b^4c^2-ay+4$

SUBTRACTION.

(Art. 4.) We do not approve of the use of the term *subtraction*, as applied to algebra, for in many cases subtraction appears like addition, and addition like subtraction. We prefer to use the expression, *finding the difference.*

What is the difference between 12 and 20 degrees of north latitude? This is subtraction. But when we demand the difference of latitude between 6 degrees north and 3 degrees south, the result *appears* like addition, for the difference is really 9 degrees, the sum of 6 and 3. This example serves to explain the true nature of the *sign minus.* It is merely an opposition to the *sign plus;* it is counting in *another direction;* and if we call the degrees north of the equator *plus*, we must call those south of it *minus*, taking the equator as the zero line.

So it is on the thermometer scale; the divisions above zero are called *plus*, those below *minus.* Money due to us may be called *plus;* money that we owe should then be called *minus*,—the one circumstance is directly opposite, in effect, to the other. *Indeed, we can conceive of no quantity less than nothing*, as we sometimes express ourselves. It is quantity in opposite circumstances or counted in an opposite direction; *hence the difference or space between a positive and a negative quantity is their apparent sum.*

As a further illustration of finding differences. let us take the following examples, which all can understand:

From	16	16	16	16	16	16
Take	12	8	2	0	—2	—4
Differ.	4	8	14	16	18	20

Here the reader should strictly observe that the smaller the number we take away, the greater the remainder, and when the subtrahend becomes minus, it must be added.

From	$12a$	$12a$	$12a$	$12a$	$12a$	$12a$
Take	$20a$	$16a$	$12a$	$9a$	$6a$	a
Diff.	$-8a$	$-4a$	0	$3a$	$6a$	$11a$

When a greater is taken from a less, we cannot have a *positive* or *plus* difference, it must be minus.

From	$20a$	$10a$	$5a$	0	0	$-5a$	$-10a$
Take	$11a$	$11a$	$11a$	$11a$	$-b$	$-b$	$-5a$
Diff.	$9a$	$-a$	$-6a$	$-11a$	$+b$	$b-5a$	$-5a$

Here it will be perceived, that any quantity subtracted from zero will be the same quantity with its sign changed.

(Art. 5.) Unlike quantities cannot be written in one sum, (Art 1,) but must be taken one after another with their proper signs; therefore, the difference of unlike quantities can only be expressed by signs. Thus the difference between a and b is $a-b$, a positive quantity if a is greater than b, otherwise it is negative From a take $b-c$, (observe that they are unlike quantities).

OPERATION.

From	$a+0+0$
Take	$0+b-c$
Remainder, or difference,	$a-b+c$

This formal manner of operation may be dispensed with; the ciphers need not be written, and the signs of the subtrahend need only be changed.

From the preceding observation, we draw the following

GENERAL RULE FOR SUBTRACTION, OR ALGEBRAIC DIFFERENCES.

Change the signs of the subtrahend, or conceive them to be changed; then proceed as in addition.

EXAMPLES.

	1.	2.	3.
From	$4a+2x-3c$	$3ax+2y$	$a+b$
Take	$a+4x-6c$	$xy-2y$	$a-b$
Remainder,	$3a-2x+3c$	$3ax-xy+4y$	$2b$

	4.	5.	6.
From	$2x^2-3x+y^2$	$7a+2-5c$	$\frac{1}{2}x+\frac{1}{2}y$
Take	$-x^2-4x+a$	$-a+2+c$	$\frac{1}{2}x-\frac{1}{2}y$
Rem.	$3x^2+x+y^2-a$	$8a\ *\ -6c$	y

	7.	8.
From	$8x^2-3xy+2y^2+c$	$ax+bx+cx$
Take	$x^2-6xy+3y^2-2c$	$x+ax+bx$
Diff.	$7x^2+3xy-y^2+3c$	$(c-1)x$

	9.	10.
From	$ax+by+cz$	$a+b+c$
Take	$-mx-ny-pz$	$-a-b-c$
Diff.	$(a+m)x+(b+n)y+(c+p)z$	$2a+2b+2c$

(Art. 6.) From a take b. The result is $a-b$. The minus sign here shows that the operation has been performed; b was positive before the subtraction; *changing the sign performed the subtraction;* so changing the sign of any other quantity would subtract it.

11. From $3a$ take $(ab+x-c-y)$, considering the terms in the vinculum as *one term*, the difference must be $3a-(ab+x-c-y)$, but if we subtract this quantity not as a whole, but term by term, the remainder must be $3a-ab-x+c+y$.

That is, when the vinculum is taken away, all the signs within the vinculum must be changed.

12. From $30xy$ take $(40xy-2b^2+3c-4d)$.
Rem. $2b^2-10xy-3c+4d$.

13. From $\sqrt{x+y}+3ax-12$ take $(4\sqrt{x+y}-2ax+b)$.
Rem. $5ax-3\sqrt{x+y}-12-b$.

14. Find the difference between $6y^2-2y-5$ and $-8y^2-5y+12$.
Ans. $14y^2+3y-17$.

15. From $3a-b-2x+7$ take $8-3b+a+4x$.
Ans. $2a+2b-6x-1$.

16. From $3p+q+r-2s$ take $q-8r+2s-8$.

Ans. $3p+9r-4s+8$

17. From $13a^2-2ax+9x^2$ take $5a^2-7ax-x^2$.

Ans. $8a^2+5ax+10x^2$

18. From $20xy-5\sqrt{a}+3y$ take $4xy+5a^{\frac{1}{2}}-y$

Ans. $16xy-10a^{\frac{1}{2}}+4y$.

19. From the sum of $6x^2y-11ax^2$, and $8x^2y+3ax^2$, take $4x^2y-4ax^2+a$. Diff. $10x^2y-4ax^2-a$

20. From the sum of $15a^2b+8cdx-3$ and $24-8a^2b+2cdx$ take the sum of $12a^2b-3cdx-8$ and $16+cdx-4a^2b$.

Diff. $12cdx+13-a^2b$

21. From the difference between $8ab-12cy$ and $-3ab+4cy$ take the sum of $5ab-7cy$ and $ab+cy$.

Diff. $5ab-10cy$.

From $2a+2b$ take $-a-b$.
From $ax+bx$ take $ax-bx$.
From $a+c+b$ take $a+c-b$.
From $3x+2y+2$ take $5x+3y+b$.
From $6a+2x+c$ take $5a+6x-3c$.
From $-4a-2x-2$ take $-6a-2x-2$.
From $12x-2xy+3$ take $7+6y+10x$.

MULTIPLICATION.

(Art. 7.) The nature of multiplication is the same in arithmetic and algebra. It is repeating one quantity as many times as there are units in another; the two quantities may be called factors, and in abstract quantities, either may be called the multiplicand; the other will of course be the multiplier.

Thus 4×5. It is indifferent whether we consider 4 repeated 5 *times* or 5 repeated 4 *times;* that is, it is indifferent which we call the multiplier. Let a represent 4, and b represent 5, then the product is $a\times b$; or with letters we may omit the sign and the product will be simply ab.

The product of any number of letters, as $a\ b\ c\ d$, is $abcd$.

The product of $x\ y\ z$ is xyz.

In the product it is no matter in what order the letters are placed; xy and yx is the same product.

The product of $ax \times by$ is $axby$ or $abxy$. Now suppose $a=6$ and $b=8$, then $ab=48$, and the product of $ax \times by$ would be the same as the product of $6x \times 8y$ or $48xy$. From this we draw the following rule for multiplying simple quantities:

Multiply the coefficients together and annex the letters, one after another, to the product.

EXAMPLES.

1. Multiply $3x$ by $7a$. *Prod.* $21ax$.
2. Multiply $4y$ by $3ab$ *Prod.* $12aby$.
3. Multiply $3b$ by $5c$, and that product by $10x$. *Prod.* $150bcx$
4. Multiply $6ax$ by $12by$ by $7ad$. *Prod.* $504aaxydb$
5. Multiply $3ac$ by $11b$ by xy.
6. Multiply af by pq by 4.

In the above examples no signs were expressed, and of course plus was understood; and it is as clear as an axiom that plus multiplied by plus must produce plus, or a positive product.

(Art. 8.) As algebraic quantities are liable to be affected by negative signs, we must investigate the products arising from them. Let it be required to multiply -4 by 3, that is, repeat the negative quantity 3 times; the whole must be negative, *as the sum of any number of negative quantities is negative.* Hence *minus multiplied by plus gives minus,** $-a \times b$ gives $-ab$; also a multiplied by $-b$ must give $-ab$, as we may conceive the minus b repeated a times.

(Art. 9.) Now let it be required to multiply -4 by -3, that is, minus 4 must be *subtracted* 3 times; but to subtract *minus* 4 is the same as to add 4, (Art. 5,) giving a positive or plus quantity; and to subtract it 3 times, as the -3 indicates, will give a product of $+12$.

That is, minus multiplied by minus gives plus.

This principle is so important that we give another mode of illustrating it:

Required the product of $a-b$ by $a-c$.

Here $a-b$ must be repeated $a-c$ times.

If we take $a-b$, a times, we shall have too large a product, as the multiplier a is to be diminished by c.

That is $a-b$

Multiplied by $\underline{a}$

Gives $aa-ab$, which is too great by $a-b$ repeated c times, or by $ac-cb$, which must be subtracted from the former product; but to subtract we change signs, (Art. 5,) therefore the true product must be $aa-ab-ac+cb$.

That is, the product of *minus* b by *minus* c gives plus bc, *and, in general, minus multiplied by minus gives plus.*

But *plus* quantities multiplied by plus give plus, and minus by plus, or plus by minus, give *minus;* therefore we may say, in short,

That quantities affected by like signs, when multiplied together, give plus, and when affected by unlike signs give minus.

(Art. 10.) The product of a into b can only be expressed by ab or ba. The product of $a\,b\,c\,d$, &c., is $abcd$; but if b c and d are each equal to a, the product would be $aaaa$.

The product of aa into aaa is $aaaaa$; but for the sake of *brevity* and *convenience*, in place of writing aaa, we write a^3. The figure on the right of the letter shows how many times the letter is taken *as a factor*, and is called an *exponent*. The product of a^3 into a^4 is a repeated 3 times as a factor, and 4 times as a factor, in all 7 times; that is, *write the letter and add the exponents.*

EXAMPLES.

What is the product of a^2 by a^5?	*Ans.* a^7.
What is the product of x^4 by x^6?	*Ans.* x^{10}.
What is the product of y^2 by y^3 by y^5?	*Ans.* y^{10}.
What is the product of a^n by a^m?	*Ans.* a^{n+m}.
What is the product of b^7x^3 by bx.	*Ans.* b^8x^4.
What is the product of ac by ac^2 by a^3c^2?	*Ans.* a^5c^5.

If adding numeral exponents is a true operation, it must be equally true when the exponents are literal.

N. B. When the exponent is not expressed, one is understood, for a is certainly the same as a^1, or once taken.

(Art. 11.) Every factor must appear or be contained in a product. Thus ax^2 multiplied by bx^3 must be abx^5. Now if $a=6$ and $b=10$ the product would be $60x^5$.

Multiply $3a^2$ by $7a^3$. Product $21a^5$.

From this we draw the following rule for the multiplication of exponential quantities.

Multiply the coefficients and add the exponents of the same letter. All the letters must appear in the product.

EXAMPLES.

Multiply $4a^4$ by $3a$. *Prod.* $12a^5$.

Multiply $3x^2$ by $-2x^3$. *Prod.* $-6x^5$.

Multiply $3x$ by $7x^2$ by $3a^2y$. *Prod.* $63a^2x^3y$.

What is the product of $2ax^2$, $4axy$, $7abx$?

Prod. $56a^3x^4by$.

What is the product of $2a^n$, $3a^mx$, and ax?

Prod. $6a^{n+m+1}x^2$.

Multiply $9a^2x$ by $4x$. *Prod.* $36a^2x^2$.

Multiply $17a^3b^2c^3$ by $7ac$. *Prod.* $119a^4b^2c^4$.

Multiply $11a^5b^2c$ by $10a^5b^8c^9$. *Prod.* $110a^{10}b^{10}c^{10}$.

Multiply $121b^2c^3x$ by $5a^4bxy^2$. *Prod.* $605a^4b^3c^3x^2y^2$.

Multiply $77a^3cx^4$ by $61a^2b$. *Prod.* $4697a^5bcx^4$.

Multiply $117ab^2c^3x$ by $2a^3b^2c$. *Prod.* $234a^4b^4c^4x$.

Multiply $9a^2x$ by $6x$.

Multiply $9ax^2$ by $-7ax$.

Multiply $7ax$ by -4.

Multiply $3ac$ by $-2cx$ by $-4c$. *Prod.* $24ac^3x$

(Art. 12.) When one compound quantity is to be multiplied or repeated as *many times as there are units in another,* it is evident that the multiplicand must be repeated by every term of the multiplier.

Thus the product of $a+b+c$ by $x+y+z$.

It is evident that $a+b+c$ must be repeated x times, then y times, then z times; and the operation may stand thus:

	$a+b+c$
	$x+y+z$
Product by x	$ax+bx+cx$
Product by y	$ay+by+cy$
Product by z	$az+bz+cz$
Entire Product	$ax+bx+cx+ay+by+cy+az+bz+cz$.

From the foregoing articles we draw the following general rule for the multiplication of compound quantities.

Multiply all the terms of the multiplicand by each term of the multiplier, observing that like signs, in both factors, give plus, and unlike, minus.

Write each term of the product distinctly by itself, with its proper sign, and afterwards condense or connect the terms as much as possible, as in addition.

EXAMPLES.

	1.	2.
Multiply	$2ax-3x$	$3x+2y$
By	$2x+4y$	$4x-5y$
Partial product	$4ax^2-6x^2$	$12x^2+8xy$
2d part. prod.	$8axy-12xy$	$-15xy-10y^2$
Whole prod.	$4ax+8axy-6x^2-12xy$	$12x^2-7xy-10y^2$

3. Multiply $2x^2+xy-2y^2$

By	$3x-3y$
Partial product	$6x^3+3x^2y-6xy^2$
2d partial product	$-6x^2y-3xy^2+6y^3$
Whole product	$6x^3-3x^2y-9xy^2+6y^3$

4. Multiply $3a^2-2ab-b^2$ by $2a-4b$.
Prod $6a^3-16a^2b+6ab^2+4b^3$.

5. Multiply x^2-xy+y^2 by $x+y$. *Prod.* x^3+y^3

6. Multiply $a^2-3ac+c^2$ by $a-c$.

Prod. $a^3-4a^2c+4ac^2-c^3$.

7. Multiply $a+b$ by $a+b$. *Prod.* $a^2+2ab+b^2$.

8. Multiply $x+y$ by $x+y$. *Prod.* $x^2+2xy+y^2$.

9. Multiply $a-b$ by $a-b$. *Prod.* $a^2-2ab+b^2$.

10. Multiply $x-y$ by $x-y$. *Prod.* $x^2-2xy+y^2$.

(Art. 13.) By inspecting all the problems, from the 7th to the 10th, we shall perceive that they are all binomial quantities, and the multiplicand and multiplier the same.

But when a number is to be multiplied into itself the product is called a *square*. Now by inspecting the products, we find that the square of any binomial quantity is equal to *plus; the squares of the two parts and twice the product of the two parts.*

N. B. The product of the two parts will be plus or minus, according to the sign between the terms of the binomial.

Let us now examine the product of $a+b$ into $a-b$.

$$\begin{array}{lll}
 & a+b & \\
 & a-b & \\
\hline
 & a^2+ab & \\
 & \quad -ab-b^2 & \\
\hline
\text{Product} & a^2 \qquad -b^2 &
\end{array}
\qquad\qquad
\begin{array}{l}
2m+2n \\
2m-2n \\
\hline
4m^2+4mn \\
\quad -4mn-4n^2 \\
\hline
4m^2 \qquad -4n^2
\end{array}$$

Multiply $2a+3b$ by $2a-3b$. *Prod.* $4a^2-9b^2$.

Multiply $3y+c$ by $3y-c$. *Prod.* $9y^2-c^2$.

Thus, by inspection, we find the product of the sum and difference of two quantities is equal to the difference of their squares.

The propositions included in this article are proved also in geometry.

(Art. 14.) We can sometimes make use of binomial quantities greatly to our advantage, as a few of the following examples will show:

1. Multiply $a+b+c$, by $a+b+c$.

Suppose $a+b$ represented by s, then it will be $s+c$.

The square of this is $s^2+2sc+c^2$; restoring the value of s, and we have $(a+b)^2+2(a+b)c+c^2$.

2. Square $x+y-z$. Let $x+y=s$.

Then $(s-z)^2=s^2-2sz+z^2=(x+y)^2-2(x+y)z+z^2$.

3. Multiply $x+y+z$ by $x+y-z$. *Prod.* $(x+y)^2-z^2$.

4. Multiply $2x^2-3x+2$ by $x-8$.

Prod. $2x^3-19x^2+26x-16$.

5. Multiply $ax+by$ by $ax+cy$.

Prod. $a^2x^2+(ab+ac)xy+cby^2$.

6. Multiply $\frac{1}{2}x+y$ by $\frac{1}{2}x-y$. *Prod* $\frac{1}{4}x^2-y^2$.

7. Multiply $a^3+2a^2b+2ab^2+b^3$

By $a^3-2a^2b+2ab^2-b^3$

$$\begin{array}{llllllll}
a^6 & +2a^5b & +2a^4b^2 & +a^3b^3 & & & & \\
 & -2a^5b & -4a^4b^2 & -4a^3b^3 & -2a^2b^4 & & & \\
 & & +2a^4b^2 & +4a^3b^3 & +4a^2b^4 & +2ab^5 & & \\
 & & & -a^3b^3 & -2a^2b^4 & -2ab^5 & -b^6 & \\
\hline
\end{array}$$

Prod. a^6-b^6

8. Multiply $x^2-\frac{1}{2}x+\frac{2}{3}$

By $\frac{1}{3}x+2$

$$\begin{array}{llll}
\frac{1}{3}x^3 & -\frac{1}{6}x^2 & +\frac{2}{9}x & \\
 & +2x^2 & -x & +\frac{4}{3} \\
\hline
\end{array}$$

Product, $\frac{1}{3}x^3+\frac{11}{6}x^2-\frac{7}{9}x+\frac{4}{3}$

9. What is the product of a^m+b^m by a^n+b^n?

Ans. $a^{m+n}+a^nb^m+a^mb^n+b^{m+n}$.

10. What is the product of $x^2-\frac{3}{4}x$ by $x^2-\frac{1}{2}x$?

Ans. $x^4-\frac{5}{4}x^3+\frac{3}{8}x^2$

11. What is the product of $4x^3+8x^2+16x+32$ by $3x-6$?

Ans. $12x^4-192$.

12. What is the product of $a^3+a^2b+ab^2+b^3$ by $a-b$?

Ans. a^4-b^4

DIVISION.

(Art. 15.) Division is the converse of multiplication, the product being called a dividend, and one of the factors a divisor. If a multiplied by b give the product ab, then ab divided by a must give b for a quotient, and if divided by b, give a. In short, if one simple quantity is to be divided by another simple quantity, the quotient must be found by *inspection*, as in division of numbers.

EXAMPLES.

1. Divide $16ab$ by $4a$. **Ans.* $4b$
2. Divide $21acd$ by $7c$. *Ans.* $3ad$
3. Divide ab^2c by ac. *Ans.* b^2.
4. Divide $3axy$ by $2bc$. *Ans.* $\frac{3axy}{2bc}$

In this last example, and in many others, the absolute division cannot be effected. In some cases it can be partially effected, and the quotients must be *fractional.*

5. Divide $3acx^2$ by acy. *Ans.* $\frac{3x^2}{y}$.
6. Divide $72b^2x$ by $8abx$. *Ans.* $\frac{9b}{a}$.
7. Divide $27aby$ by $11abx$. *Ans.* $\frac{27y}{11x}$.

(Art. 16.) It will be observed that the product of the divisor and quotient must make the dividend, and the signs must conform to the principles laid down in multiplication. The following examples will illustrate :

8. Divide $-9y$ by $3y$. *Ans* -3.
9. Divide $-9y$ by $-3y$. *Ans.* $+3$
10. Divide $+9y$ by $-3y$. *Ans.* -3.

---

* The term quotient would be more exact and technical here ; but, in results hereafter, we shall invariably use the term *Ans.*, as more brief and elegant, and it is equally well understood.

(Art. 17.) The product of a^3 into a^2 is a^5, (Art. 10,) that is, in multiplication we add the exponents; and as division is the converse of multiplication, to divide powers of the *same letter*, we must *subtract the exponent of the divisor from that of the dividend.*

Divide $2a^6$ *by* a^4. *Ans.* $2a^2$.

Divide $-a^7$ by a^6. *Ans.* $-a$.

Divide $16x^3$ by $4x$. *Ans.* $4x^2$.

Divide $15axy^3$ by $-3ay$. *Ans.* $-5xy^2$.

Divide $63a^m$ by $7a^n$. *Ans.* $9a^{m-n}$.

Divide $12ax^n$ by $-3ax$. *Ans.* $-4x^{n-1}$.

Divide $7a^2b$ by $21a^3b^2$. *Ans.* $\frac{7a^2b}{21a^3b^2}=\frac{1}{3ab}$.

Divide $-5a^2x^2$ by $-7a^4x^2$. *Ans.* $\frac{5}{7a^2}$.

Divide $117a^5b^4c^3$ by $78a^5bc^4$. *Ans.* $\frac{3b^3}{2c}$.

Divide $96abc$ by $12a^3bc^2d$. *Ans.* $\frac{8}{a^2cd}$.

Divide a^5bc^2 by $a^6b^2c^4$. *Ans.* $\frac{1}{abc^2}$.

Divide $27a^3b^4cd^2$ by $21abcd$. *Ans.* $\frac{9}{7}a^2b^3d$.

Divide $14ab^2cd$ by $6a^2bc^2$. *Ans.* $\frac{7bd}{3ac}$.

(Art. 18.) The object of this article is to explain the nature of negative exponents.

Divide a^4 successively by a, and we shall have the following quotients:

$$a^3, \quad a^2, \quad a, \quad 1, \quad \frac{1}{a}, \quad \frac{1}{a^2}, \quad \frac{1}{a^3}, \quad \&c.$$

Divide a^4 again, *rigidly adhering* to the principle that to divide any power of a by a, the exponent becomes *one less*, and we have

$$a, \quad a^2, \quad a^1, \quad a^0, \quad a^{-1}, \quad a^{-2}, \quad a^{-3}, \quad \&c.$$

Now these quotients must be equal, that is, a^3 in one series equals a^3 in the other, and

$$a^2=a^2, \quad a=a^1, \quad 1=a^0, \quad \frac{1}{a}=a^{-1} \quad \frac{1}{a^2}=a^{-2} \quad \frac{1}{a^3}=a^{-3}$$

Another illustration. We divide exponential quantities by subtracting the exponent of the divisor from the exponent of the dividend. Thus a^5 divided by a^2 gives a quotient of $a^{5-2}=a^3$. a^5 divided by $a^7=a^{5-7}=a^{-2}$. We can also divide by taking the dividend for a numerator and the divisor for a denominator, thus

$\frac{a^5}{a^7}=\frac{1}{a^2}$, therefore $\frac{1}{a^2}=a^{-2}$ (Axiom 7.)

From this we learn, *that exponential factors may be changed from a numerator to a denominator, and the reverse, by changing the signs of the exponents.*

Thus, $\frac{a}{x^2}=ax^{-2} \qquad \frac{a^{-3}}{3y^{-1}}=\frac{y}{3a^3} \qquad \frac{x^m}{x^n}=x^{m-n}$

Divide a^5bc by $a^6b^2c^{-1}$. *Ans.* $a^{-1}b^{-1}c^2$.

Observe, that to divide is to subtract the exponents.

Divide $14ab^2cd$ by $6a^2bc^2$. *Ans.* $\frac{7bd}{3ac}=\frac{7}{3}bda^{-1}c^{-1}$.

(Art. 19.) A compound quantity divided by a simple quantity, is effected by dividing each term of the compound quantity by the simple divisor.

EXAMPLES.

1. Divide $3ax-15x$ by $3x$. *Ans.* $a-5$.
2. Divide $8x^2+12x^3$ by $4x^2$. *Ans.* $2x+3$.
3. Divide $3bcd+12bcx-9b^2c$ by $3bc$. *Ans.* $d+4x-3b$.
4. Divide $7ax+3ay-7bd$ by $-7ad$.

Ans. $-\frac{x}{d}-\frac{3y}{7d}+\frac{b}{a}$

5. Divide $15a^2bc—15acx^2+5ad^2$ by $—5ac$.

Ans. $—3ab+3x^2—\frac{d^2}{c}$.

6. Divide $10x^3—15x^2—25x$ by $5x$. *Ans.* $2x^2—3x—5$.

7 Divide $—10ab+60ab^3$ by $—6ab$.

Ans. $\frac{5}{3}—10b^2$.

8. Divide $36a^2b^2+60a^2b—6ab$ by $—12ab$.

Ans. $—3ab—5a+\frac{1}{2}$.

9. Divide $10rx—cry+2crx$ by cr.

10. Divide $10vy+16d$ by $2d$.

11. Divide $6ay—18acd+24a$ by $6a$.

12. Divide $mx—amx+m$ by m.

(Art. 20.) We now come to the last and most important operation in division, the division of one compound quantity by another compound quantity.

The dividend may be considered a *product* of the divisor into the yet unknown factor, the quotient; and the highest power of any letter in the product, or the now called dividend, must be conceived to have been formed by the highest power of the same letter in the divisor into the highest power of that letter in the quotient. *Therefore, both the divisor and the dividend must be arranged according to the regular powers of some letter.*

After this, the truth of the following rule will become obvious by its great similarity to division in numbers.

RULE. *Divide the first term of the dividend by the first term of the divisor, and set the result in the quotient.*[*]

Multiply the whole divisor by the quotient thus found, and subtract the product from the dividend.

The remainder will form a new dividend, with which proceed as before, till the first term of the divisor is no longer contained in the first term of the remainder.

The divisor and remainder, if there be a remainder, are then

---

[*] Divide the *first term* of the dividend and of the remainders by the *first term* of the divisor; be not troubled about other terms.

to be written in the form of a fraction, as in division of numbers.

EXAMPLES.

Divide $a^2+2ab+b^2$ by $a+b$.

Here, a is the leading letter, and as it stands first in both the dividend and divisor, no change of place is necessary.

OPERATION.

$$\begin{array}{l}
a+b)a^2+2ab+b^2(a+b \\
\quad\;\; \underline{a^2+ab\qquad\quad} \\
\qquad\qquad ab+b^2 \\
\qquad\qquad \underline{ab+b^2}
\end{array}$$

Agreeably to the rule, we consider that a will be contained in a^2, a times; then the product of a into the divisor is a^2+ab, and the *first* term of the remainder is ab, in which a is contained b times. We then multiply the divisor by b, and there being no remainder, $a+b$ is the whole quotient.

Divide $a^3+3a^2x+3ax^2+x^3$ by $x+a$.

As the highest power of a stands in the first term of the dividend, and the powers of a decrease in regular gradation from term to term, therefore we must change the terms of the divisor to make a stand first.

OPERATION.

1.

$$\begin{array}{l}
a+x)a^3+3a^2x+3ax^2+x^3(a^2+2ax+x^2 \\
\quad\;\; \underline{a^3+a^2x\qquad\quad} \\
\qquad\quad 2a^2x+3ax^2 \\
\qquad\quad \underline{2a^2x+2ax^2\qquad} \\
\qquad\qquad\qquad ax^2+x^3 \\
\qquad\qquad\qquad \underline{ax^2+x^3}
\end{array}$$

2.
$$\begin{array}{l}
a-c)a^3-4a^2c+4ac^2-c^3(a^2-3ac+c^2 \\
\quad a^3-\ a^2c \\
\hline
\qquad -3a^2c+4ac^2 \\
\qquad -3a^2c+3ac^2 \\
\hline
\qquad\qquad\quad ac^2-c^3 \\
\qquad\qquad\quad ac^2-c^3 \\
\hline
\end{array}$$

3.
$$\begin{array}{l}
a^2-4a+4)a^3-6a^2+12a-8(a-2 \\
\qquad a^3-4a^2+\ 4a \\
\hline
\qquad\quad -2a^2+\ 8a-8 \\
\qquad\quad -2a^2+\ 8a-8 \\
\hline
\end{array}$$

4. Divide $6x^4-96$ by $6x-12$. *Ans.* x^3+2x^2+4x+8.

5. Divide a^2-b^2 by $a-b$. *Ans.* $a+b$.

6. Divide $25x^6-x^4-2x^3-8x^2$ by $5x^3-4x^2$.
Ans. $5x^3+4x^2+3x+2$.

(Art. 21.) We may cast out equal factors from the dividend and divisor, without changing the value of the quotient; for $amxy$ divided by am gives xy for a quotient; cast out either of the common factors a or m from both dividend and divisor, and we shall still have xy for a quotient. This, in many instances, will greatly facilitate the operation. Thus, in the 4th example, the factor 6 may be cast out, as it is contained in all the terms; and in the 6th example the factor x^2 may be cast out; the quotients will of course be the same.

7. Divide $a^2+4ax+4x^2+y^2$ by $a+2x$.

Ans. $a+2x+\dfrac{y^2}{a+2x}$

8. Divide $6a^4+9a^2-15a$ by $3a^2-3a$.
(Observe Art. 21.) *Ans.* $2a^2+2a+5$.

9. Divide x^6-y^6 by $x^3+2x^2y+2xy^2+y^3$.
Ans. $x^3-2x^2y+2xy^2-y^3$.

10. Divide $ax^3-(a^2+b)x^2+b^2$ by $ax-b$.

Ans. x^2-ax-b

11. Divide 1 by $1-a$. *Ans.* $1+a+a^2+a^3$, &c., &c.

12. Divide $x^3+\frac{3x^2}{4}+\frac{3x}{4}+1$ by $\frac{x}{2}+\frac{1}{2}$.

Ans. $2x^2-\frac{x}{2}+2$

N. B. We may multiply both dividend and divisor by the same number as well as divide them.

13. Divide $1-5y+10y^2-10y^3+5y^4-y^5$ by $1-2y+y^2$.

Ans. $1-3y+3y^2-y^3$

14. Divide a^4+4b^4 by $a^2-2ab+2b^2$.

Ans. $a^2+2ab+2b^2$.

15. Divide $x^6-x^4+x^3-x^2+2x-1$ by x^2+x-1.

Ans. $x^4-x^3+x^2-x+1$.

16. Divide a^5-x^5 by $a-x$.

Ans. $a^4+a^3x+a^2x^2+ax^3+x^4$.

17. Divide b^5+y^5 by $b+y$.

Ans. $b^4-b^3y+b^2y^2-by^3+y^4$.

18. Divide $a^3+5a^2y+5ay^2+y^3$ by $a+y$.

Ans. $a^2+4ay+y^2$.

If more examples are desired for practice, the examples in multiplication may be taken. The product or answer may be taken for a dividend, and either one of the factors for a divisor; the other will be the quotient.

Also, the examples in division may be changed to examples in multiplication; and these changes may serve to impress on the mind of the pupil the close connection between these two operations.

(Art. 22.) In the following examples the dividends and divisors are given in the form of fractions, and the quotients are the terms after the sign of equality. Let the pupil actually divide, and observe the quotients attentively.

1. $\frac{x^2-a^2}{x-a}=x+a.$

2. $\dfrac{x^3-a^3}{x-a}=x^2+ax+a^2.$

3. $\dfrac{x^4-a^4}{x-a}=x^3+ax^2+a^2x+a^3.$

4. $\dfrac{x^5-a^5}{x-a}=x^4+ax^3+a^2x^2+a^3x+a^4.$

Hence we may conclude that in general x^m-a^m is divisible by $x-a$, m being any entire positive number.

That is, $\dfrac{x^m-a^m}{x-a}=x^{m-1}+ax^{m-2}+\cdots a^{m-2}x+a^{m-1}$

The quotient commencing with a power of x, *one less* than m, and ending with a power of a, one less than m.

These divisions show, that the difference of two equal powers of different quantities is always divisible by the difference of their roots.

(Art. 23.) By trial, that is, actual division, we shall find that

$$\frac{x^2-a^2}{x+a}=x-a.$$

$$\frac{x^4-a^4}{x+a}=x^3-ax^2+a^2x-a^3.$$

$$\frac{x^6-a^6}{x+a}=x^5-ax^4+a^2x^3-a^3x^2+a^4x-a^5.$$

&c. &c. &c. &c.

From which we learn that the difference of any two equal powers of different quantities, is also divisible by the sum of their roots when the exponent of the power is an even number.

(Art. 24.) By actual division we find that

$$\frac{x^3+a^3}{x+a}=x^2-ax+a^2.$$

$$\frac{x^5+a^5}{x+a}=x^4-ax^3+a^2x^2-a^3x+a^4.$$

And in general, we may conclude that the sum of any two equal powers of different quantities, is divisible by the sum of their roots when the exponent of the power is an *odd* number.

In Art. 22, if we make $a=1$ the formulas become

$$\frac{x^2-1}{x-1}=x+1.$$

$$\frac{x^3-1}{x-1}=x^2+x+1.$$

$$\frac{x^m-1}{x-1}=x^{m-1}+x^{m-2}+x^{m-3}, \text{ \&c.}$$

If we make $x=1$, what will the formulas become?

Make the same substitutions in articles 23 and 24, and examine the results.

By inspecting articles 22, 23, and 24, we find that

$$(x+a)(x-a)=x^2-a^2, \quad (x^2+ax+a^2)(x-a)=x^3-a^3, \quad \text{\&c.},$$

for the product of the divisor and quotient must always produce the dividend. These principles point out an expedient of condensing a multitude of terms by multiplying them by the roots of the terms involved. Thus, $x^4 \pm ax^3+a^2x^2 \pm a^3x+a^4$, can be condensed to two terms by multiplying them by $x \pm a$, the root of the first and last term, with the minus sign where the signs are plus in the multiplicand, and with the plus sign where the signs are alternately plus and minus. See examples in Art. 22 and 24.

ALGEBRAIC FRACTIONS.

(Art. 25.) We shall be very brief on the subject of algebraic fractions, because the names and rules of operations are the same as numeral fractions in common arithmetic; and for illustration, shall, in some cases, place them side by side.

Case i. *To reduce a mixed quantity to an improper fraction, multiply the integer by the denominator of the fraction, and to the product add the numerator, or connect it with its proper sign, + or —; then the denominator being set under this sum, will give the improper fraction required.*

EXAMPLES.

1. Reduce $2\frac{3}{8}$ and $a+\frac{x}{b}$ to improper fractions.

Ans. $\frac{19}{8}$ and $\frac{ab+x}{b}$

These two operations, and the principle that governs them, are exactly alike.

2. Reduce $5\frac{7}{8}$ and $a+\frac{a^2}{b}$ to improper fractions.

Ans. $\frac{47}{8}$ and $\frac{ab+a^2}{b}$

3. Reduce $4-\frac{1}{3}$ and $a-\frac{b}{x}$ to improper fractions.

Ans. $\frac{11}{3}$ and $\frac{ax-b}{x}$

4. Reduce $5-\frac{4-1}{5}$ and $2b-\frac{3x-a}{c}$ to improper fractions.

Ans. $\frac{25-4+1}{5}=\frac{22}{5}$ and $\frac{2bc-3x+a}{c}$

5. Reduce $5a+\frac{ab+x}{b}$ to an improper fraction.

6. Reduce $12+\frac{3a+b}{b}$ to an improper fraction.

7. Reduce $4+2x+\frac{b}{c}$ to an improper fraction.

8. Reduce $5x-\frac{2x-5}{3}$ to an improper fraction.

9. Reduce $3a-9-\frac{3a^2-30}{a+3}$ to an improper fraction.

Ans. $\frac{3}{a+3}$

CASE 2. *The converse of Case* 1. *To reduce improper fractions to mixed quantities, divide the numerator by the denominator, as far as possible, and set the remainder, (if any,)* *over the denominator for the fractional part; the two joined together with their proper sign, will be the mixed quantity sought.*

EXAMPLES.

1. Reduce $\frac{47}{8}$ and $\frac{ab+x}{b}$ to mixed quantities.

Ans. $5\frac{7}{8}$ and $a+\frac{x}{b}$

2. Reduce $\frac{19}{8}$ and $\frac{a^2+bx}{a}$ to mixed quantities.

Ans. $2\frac{3}{8}$ and $a+\frac{bx}{a}$

3. Reduce $\frac{5ay+ab+y}{y}$ to a mixed quantity.

Ans. $5a+\frac{ab+y}{y}$.

4. Reduce $\frac{2a^2-2b^2}{a-b}$ to a whole or mixed quantity.

Ans. $2a+2b$

5. Reduce $\frac{2x^3-2y^3}{x-y}$ to a whole number.

Ans. $2(x^2+xy+y^2)$ by (Art. 22.)

6. Reduce $\frac{4c+2cx+b}{c}$ to a mixed quantity.

7. Reduce $\frac{10a^2-4a+6}{5a}$ to a mixed quantity.

8. Reduce $\frac{13x+5}{3}$ to a mixed quantity.

9. Reduce $\frac{3x^2-12ax+y-9x}{3x}$ to a mixed quantity.

(Art. 26.) It is very desirable to obtain algebraic quantities in their most condensed form. Therefore, it is often necessary to reduce fractions to their lowest terms; and this can be done as in arithmetic, by dividing both numerator and denominator by their obvious common factors, or for their final reduction, by their greatest common measure. If the terms have no common measure, the fraction is already to its lowest terms.

The principle on which these reductions rest is that of division, explained in (Art. 21).

CASE 3. *To find the greatest common measure of the terms of a fraction, divide the greater term by the less, and the last divisor by the remainder, and so on till nothing remains; then the divisor last used will be the common measure required.*

But note, that it is proper to arrange the quantities according to the powers of some letter, as is shown in division.

N. B. During the operation we may cast out, or throw in a factor to either one of the terms which is not a factor in the other, as such a factor would make no part of the common measure, and the *value* of quantities is not under consideration. Thus, the fraction $\frac{ab+b^2}{a^2-b^2}$ has $a+b$ for its greatest common measure; and this quantity is not affected by casting out the factor b from the numerator, and seeking the common measure of the fraction $\frac{a+b}{a^2-b^2}$.

(Art. 27.) To demonstrate the truth of the rule for finding the greatest common measure, let us suppose D to represent a dividend, and d a divisor, q the first quotient and r the first remainder.

In short, let us represent successive divisions as follows:

$$\begin{array}{l}
d)D(q \\
\quad dq \\
\hline
\quad r)d(q' \\
\quad\quad rq' \\
\hline
\quad\quad r')r(q'' \\
\quad\quad\quad r'q'' \\
\hline
\quad\quad\quad 0
\end{array}$$

Now, in division, the dividend is always equal to the product of the divisor and quotient, plus the remainder, if any.

Therefore, $r=r'q''$

and $d=rq'+r'$

and $D=dq+r.$

As $r=r'q''$, the last divisor r' is a factor in r (there being no remainder); that is, r' *measures* r.

Now as r' measures r, it measures any number of times r, or $rq'+r'$, or d; therefore r' measures d.

Again, as r' measures d and r, it measures any number of times $d+r$; that is, it measures $dq+r$ or D.

Hence r', the last divisor, is a common measure to both D and d, or of the terms of the fraction $\frac{D}{d}$.

We have now to show that r' is not only the common measure of D and d, but the *greatest* common measure.

In division, if we subtract the product of the divisor and quotient from the dividend, we shall have the remainder.

That is, $D-dq=r$, and $d-rq'=r'$.

Now, every common measure of D and d is also a measure of r, because $D-dq=r$; for the same reason every common measure of r and d is also a measure of r'. But the *greatest* measure of r' is itself. This final remainder is, therefore, the greatest common measure of D and d.

EXAMPLES.

1. Find the greatest common measure of the two terms of the fraction $\frac{a^4-1}{a^5+a^3}$ and with it, reduce the fraction to its lowest terms.

CONSIDERATION AND OPERATION.

The denominator has a^3 as a factor to all its terms, *which is not a factor in the numerator;* hence this can form no part of the common measure, or the common measure will still be there if this factor is taken away.

We then seek the common measure of a^4-1 and a^2+1.

$$\begin{array}{l}
a^2+1)a^4 \qquad -1 \\
\qquad\quad +a^2 \\
\hline
\qquad\quad -a^2-1 \\
\qquad\quad -a^2-1 \\
\hline
\end{array}$$

Hence a^2+1 is the common measure, which, used as a divisor to both numerator and denominator, reduces the fraction to $\frac{a^2-1}{a^3}$.

2. Find the greatest common measure, and reduce the fraction $\frac{x^3-y^3}{x^4-x^2y^2}$.

$$\begin{array}{l}
x^2-y^2)x^3-y^3(x \\
\qquad\quad\; x^3-xy^2 \\
\hline
\qquad\qquad\; xy^2-y^3
\end{array}$$

Divide this rem. by y^3

$$\begin{array}{l}
x-y)x^2-\;y^2(x+y \\
\qquad\;\; x^3-xy \\
\hline
\qquad\qquad xy-y^2 \\
\qquad\qquad xy-y^2 \\
\hline
\end{array}$$

Ans. Common measure $x-y$.

Fraction reduced $\frac{x^2+xy+y^2}{x^3+x^2y}$.

3. Find the greatest common measure and reduce the fraction $\frac{a^4+2a^2x^2y+x^4y^2}{5a^3xy+5ax^3y^2}$.

$$\begin{array}{l}
a^2+x^2y)a^4+2a^2x^2y+x^4y^2(a^2+x^2y \\
\qquad\qquad a^4+\;a^2x^2y \\
\hline
\qquad\qquad\qquad a^2x^2y+x^4y^2 \\
\qquad\qquad\qquad a^2x^2y+x^4y^2 \\
\hline
\end{array}$$

Ans. Greatest common measure a^2+x^2y. Reduced fraction $\frac{a^2+x^2y}{5axy}$.

Find the greatest common measure of $a^3+3a^2b+3ao^2+b^3$ and $a^2c+2abc+b^2c$.

Reject the common factor c in one of the quantities,

$$\begin{array}{r}
a^2+2ab+b^2)\,a^3+3a^2b+3ab^2+b^3\,(a+b \\
\underline{a^3+2a^2b+\ \ ab^2} \\
a^2b+2ab^2+b^3 \\
\underline{a^2b+2ab^2+b^3}
\end{array}$$

4. Find the greatest common divisor, and reduce the fraction $\dfrac{3a^2-2a-1}{4a^3-2a^2-3a+1}$ to its lowest terms.

Here we find that neither term is divisible by the other; but if these quantities have a common divisor, such divisor will still exist if we multiply one of the terms by *any number* whatever, to render division possible.

$$\begin{array}{lr}
\text{Therefore take} & 4a^3-2a^2-\ 3a+\ 1 \\
\text{Multiply it by} & \underline{3\qquad\qquad\qquad} \\
 & 3a^2-2a-1)\,12a^3-6a^2-\ 9a+\ 3\,(4a \\
 & \underline{12a^3-8a^2-\ 4a\qquad} \\
 & 2a^2-\ 5a+\ 3 \\
\text{Multiply by} & \underline{3\qquad\qquad\quad} \\
 & 3a^2-2a-1)\,6a^2-15a+\ 9\,(2 \\
 & \underline{6a^2-\ 4a-\ 2} \\
\text{Divide by } -11 & \underline{-11a+11} \\
 & a-1)\,3a^2-2a-1\,(3a+1 \\
 & \underline{3a^2-3a\qquad} \\
 & a-1 \\
 & \underline{a-1}
\end{array}$$

Ans Greatest common measure $a-1$. Reduced fraction $\dfrac{3a+1}{4a^2+2a-1}$.

5. Find the greatest common measure and reduce the fraction $\dfrac{a^4-x^4}{a^3-a^2x-ax^2+x^3}$ to its lowest terms.

Ans. Common divisor a^2-x^2. Reduced fraction $\dfrac{a^2+x^2}{a-x}$.

6. Find the greatest common measure, and reduce the fraction $\dfrac{a^4-y^4}{a^6-y^6}$ to its lowest terms.

Ans. Common divisor a^2-y^2. Red. frac. $\dfrac{a^2+y^2}{a^4+a^2y^2+y^4}$.

7. Reduce $\dfrac{a^3-3a^2x+3ax^2-x^3}{a^2-x^2}$ to its lowest terms.

Ans. $\dfrac{a^2-2ax+x^2}{a+x}$.

8. Reduce $\dfrac{6a^2+7ax-3x^2}{6a^2+11ax+3x^2}$ to its lowest terms.

Ans. $\dfrac{3a-x}{3a+x}$

9. Reduce $\dfrac{a^3bx^3+2abx^5}{a^4x^2+2a^2x^4}$ to its lowest terms. *Ans.* $\dfrac{bx}{a}$

10. Reduce $\dfrac{6a^2x+9ax^2-12ax-8a^2}{6ax-8a}$ to its lowest terms.

Ans. $\dfrac{2a^2+3x}{2}$.

11. What is the value of $\dfrac{a^4-b^4}{a^2+b^2}$? *Ans.* a^2-b^2

12. Find the greatest common divisor of $12a^4-24a^3b+12a^2b^2$, and $8a^3b^2-24a^2b^3+24ab^4-8b^5$. *Ans.* $4(a^2-2ab+b^2.)$

(Art. 28.) We may often reduce a fraction by separating both numerator and denominator into obvious factors, without the formality of finding the greatest common divisor. The following are some examples of the kind:

1. Reduce $\frac{a^3-ab^2}{a^2+2ab+b^2}$ to its lowest terms.

$$\frac{a^3-ab^2}{a^2+2ab+b^2}=\frac{a(a^2-b^2)}{(a+b)(a+b)}=\frac{a(a-b)(a+b)}{(a+b)(a+b)}=\frac{a^2-ab}{a+b}.$$

2. Reduce $\frac{x^5-b^2x^3}{x^4-b^4}$ to its lowest terms. *Ans.* $\frac{x^3}{x^2+b^2}$.

3. Reduce $\frac{x^2-1}{xy+y}$ to its lowest terms. *Ans.* $\frac{x-1}{y}$.

4. Reduce $\frac{cx+cx^2}{acx+abx}$ to its lowest terms. *Ans.* $\frac{c+cx}{ac+ab}$.

5. Reduce $\frac{2x^2-16x-6}{3x^2-24x-9}$ to its lowest terms. *Ans.* $\frac{2}{3}$.

(Art. 29.) *To find the least common multiple of two or more quantities.*

The least common multiple of several quantities is the least quantity in which each of them is contained without a remainder.

Thus, the least common multiple of the prime factors, a, b, c, x, is obviously their product $abcx$. Now observe that the same product is the least common multiple also, when either one of these letters appears in more than one of the terms. Take a, for example, and let it appear with b, c, or x, or with all of them, as a, ab, c, ax, or a, b, ac, ax, the product $abcx$ is still divisible by each quantity. Therefore, when the same factor appears in any number of the terms, it is only necessary that it should appear once in the product; that is, once in the least common multiple. If it should be used more than once, the product so formed would not be the *least common multiple.*

From this examination, the following rule for finding the least common multiple will be obvious:

Rule. *Write the given quantities, one after another, and draw a line beneath them. Then divide by any prime factor that will divide two or more of them without a remainder, bringing down the quotients and the quantities not divisible, to a line below. Divide this second line as the first, forming* *a third, &c., until nothing but prime quantities are left. Then multiply all the divisors and the remainders that are not divisible, and their product will be the least common multiple.*

N. B. This rule is also in common arithmetic.

EXAMPLES.

1. Required the least common multiple of $8ac$, $4a^2$, $12ab$ $8ac$, and cx.

$$\begin{array}{rcccc} 2a)8ac & 4a^2 & 12ab & 8ac & cx \\ \hline 2c)4c & 2a & 6b & 4c & cx \\ \hline 2)2 & a & 3b & 2 & x \\ \hline 1 & a & 3b & 1 & x \end{array}$$

Therefore $2a \times 2c \times 2 \times a \times 3b \times x = 24a^2cbx$.

Here the divisor $2c$ will not divide $2a$, but the coefficient of c will divide the coefficient of a, and we let them divide, for it is the same as first dividing by 2, and afterwards by c. From the same consideration we permit $2c$ to divide cx, or let the letter c in the divisor strike out c before x.

By the rule we should divide by 2 and by c separately; but this is a practical abbreviation of the rule.

2. Required the least common multiple of $27a$, $15b$, $9ab$, and $3a^2$. *Ans.* $135a^2b$.

3. Find the least common multiple of (a^2-x^2), $4(a-x)$, and $(a+x)$. *Ans.* $4(a^2-x^2)$.

4. Find the least common multiple of ax^2, bx, acx, and a^2-x^2. *Ans.* $(a^2-x^2)acbx^2$.

5. Find the least common multiple of $a+b$, $a-b$, a^2+ab+b^2, and a^2-ab+b^2. *Ans.* a^6-b^6.

The least common multiple is useful many times in reducing fractions to their least common denominator.

CASE 4. *To reduce fractions to a common denominator.*

(Art. 30.) The rule for this operation, and the principle on which it is founded, is just the same as in common arithmetic, merely the multiplication of numerator and denominator by the same quantity. The object of reducing fractions to a common denominator is to *add them*, or to take their difference, as different *denominations* cannot be put into one sum.

Rule. *Multiply each numerator by all the denominators, except its own, for a new numerator, and all the denominators for a common denominator.*

Or, find the least common multiple of the given denominators for a common denominator; then multiply each denominator by such a quantity as will give the common denominator, and multiply each numerator by the same quantity by which its denominator was multiplied.

EXAMPLES.

1. Reduce $\frac{2a}{x}$ and $\frac{3b}{2c}$ to a common denominator.

Ans. $\frac{4ac}{2cx}$ and $\frac{3bx}{2cx}$.

2. Reduce $\frac{2a}{b}$ and $\frac{3a+2b}{2c}$ to a common denominator.

Ans. $\frac{4ac}{2bc}$ and $\frac{3ab+2b^2}{2bc}$.

3. Reduce $\frac{5a}{3x}$ and $\frac{3b}{2c}$, and $4d$ to a common denominator.

Ans. $\frac{10ac}{6cx}$ and $\frac{9bx}{6cx}$ and $\frac{24cdx}{6cx}$.

4. Reduce $\frac{a}{b}$, $\frac{x+1}{c}$, $\frac{y}{x+a}$, to fractions having a common denominator.

Ans. $\frac{acx+a^2c}{bcx+abc}$, $\frac{(bx+b)(x+a)}{bcx+abc}$, $\frac{bcy}{bcx+abc}$.

(Art. 31.) **Case 5** *Addition or finding the sum of fractions.*

Rule. *Reduce the fractions to a common denominator; and the sum of the numerators, written over the common denominator, will be the sum of the fractions.*

EXAMPLES.

1 Add $\frac{3x}{5}$, $\frac{2x}{7}$, and $\frac{x}{3}$ together

Ans. $\frac{63x+30x+35x}{105}=\frac{128x}{105}$

2. Add $\frac{a}{b}$ and $\frac{a+b}{c}$. *Ans.* $\frac{ac+ab+b^2}{bc}$

3 Add $\frac{1}{2}$, $\frac{a^2}{3}$ and $\frac{a^2+x^2}{a+x}$.

Ans. $\frac{3a+3x+2a^3+2a^2x+6a^2+6x^2}{6(a+x)}$.

4. Add $\frac{a+b}{a-b}$ and $\frac{a-b}{a+b}$. *Ans.* $\frac{2a^2+2b^2}{a^2-b^2}$.

5. Add $2a+\frac{a+3}{5}$ and $4a+\frac{2a-5}{4}$ *Ans.* $6a+\frac{14a-13}{20}$.

6. Add $a-\frac{8x^2}{b}$ and $b+\frac{2ax}{c}$ *Ans.* $a+b+\frac{2abx-8cx^2}{bc}$.

7. Add $5x+\frac{x-2}{3}$ and $4x-\frac{2x-3}{5x}$.

Ans. $9x+\frac{5x^2-16x+9}{15x}$.

8. What is the sum of $2b+\frac{3x}{5}$, $\frac{b}{b-x}$ and $\frac{b-x}{b}$?

Ans. $2b+2+\frac{3b^2x-3bx^2+5x^2}{5b^2-5bx}$.

9 What is the sum of $5y+\frac{y-2}{3}$ and $4y-\frac{2y-3}{5y}$?

Ans. $9y+\frac{5y^2-16y+9}{15y}$.

10. What is the sum of $5a$, $\frac{2x}{3a^2}$ and $\frac{x+2a}{4a}$?

Ans. $5a+\frac{8x+3ax+6a^2}{12a^2}$.

(Art. 32.) Case 6. *Subtraction or finding difference.*

Rule. *Reduce the fractions to a common denominator, and subtract the numerator of that fraction which is to be subtracted from the numerator of the other, placing the difference over the common denominator.*

EXAMPLES.

1. From $\frac{7x}{2}$ take $\frac{2x-1}{3}$. *Ans.* $\frac{21x-4x+2}{6}=\frac{17x+2}{6}$.

2. From $\frac{1}{x-y}$ take $\frac{1}{x+y}$. Eq. fractions $\frac{x+y}{x^2-y^2}$, $\frac{x-y}{x^2-y^2}$.

   Difference or *Ans.* $\frac{2y}{x^2-y^2}$.

3. From $\frac{x}{3}$ take $\frac{2x}{7}$. Diff. $\frac{x}{21}$.

4. From $\frac{3x}{7}$ take $\frac{2x}{9}$. *Ans.* $\frac{13x}{63}$.

5 From $\frac{2a-b}{4c}$ subtract $\frac{3a-4b}{3b}$.

Ans. $\frac{6ab-3b^2-12ac+16bc}{12bc}$.

6. From $3a+\frac{11a-10}{15}$ subtract $2a+\frac{3a-5}{7}$.

Ans. $a+\frac{32a+5}{105}$.

7. From $x+\frac{x-y}{x^2+xy}$ subtract $\frac{x+y}{x^2-xy}$. *Ans.* $x-\frac{4y}{x^2-y^2}$.

8. From $\frac{a-b}{2c}$ take $\frac{2b-4a}{5d}$. *Ans.* $\frac{5ad-5bd-4bc+8ac}{10cd}$.

9. From $3x+\frac{x}{b}$ take $x-\frac{x-a}{c}$. *Ans.* $2x+\frac{cx+bx-ab}{bc}$.

10. Find the difference between $\frac{a+b}{a-b}$ and $\frac{a-b}{a+b}$.

Ans. $\frac{4ab}{a^2-b^2}$

11 From $\frac{(x+y)^2}{xy}$ take $\frac{(x-y)^2}{xy}$ *Ans.* 4.

Case 7. *Multiplication of fractions.*

(Art. 33.) The multiplication of algebraic fractions is just the same in principle and in fact, as in numeral fractions, hence the rule must be the same.

It is perfectly obvious, that $\frac{2}{7}$ multiplied by 2 must be $\frac{4}{7}$, and multiplied by 3 must be $\frac{6}{7}$; and the result would be equally obvious with any other simple fraction; hence, to multiply a fraction by a whole number, we must multiply its numerator.

It is manifest that doubling a denominator without changing its numerator halves a fraction, thus $\frac{1}{2}$; double the 2, and we have $\frac{1}{4}$; the half of the first fraction.

Also $\frac{3}{5}$, double the 5 gives $\frac{3}{10}$, the half of $\frac{3}{5}$. In the same manner, to divide a fraction by 3 we would multiply its denominator by 3, &c. *In general, to divide a fraction by any number, we must multiply the denominator by that number.*

Now let us take the literal fraction $\frac{a}{b}$, and multiply it by c, the product must be $\frac{ac}{b}$.

Again, let it be required to multiply $\frac{a}{b}$ by $\frac{c}{d}$. Here the multiplication is the same as before, except the multiplier c is divided by d; therefore if we multiply by c we must divide by d. But the product of $\frac{a}{b}$ by c is $\frac{ac}{b}$; this must be divided by d, and we shall have $\frac{ac}{bd}$ for the true product of $\frac{a}{b}$ by $\frac{c}{d}$.

From the preceding investigation we draw the following rule to multiply fractions:

RULE. *Multiply the numerators together for a new numerator, and the denominators together for a new denominator.*

N. B. When equal factors, whether numeral or literal, appear in numerators and denominators, they may be canceled, or left out, which will save subsequent reductions.

EXAMPLES.

1. Multiply $\frac{a}{b}$ by $\frac{b}{x}$ and $\frac{ay}{c}$. *Ans.* $\frac{a^2y}{cx}$.

In this example, b in the denominator of one fraction cancels b in the numerator of another.

2. Multiply $\frac{(a+x)}{30}$ by $\frac{5a}{3(a+x)}$ *Ans.* $\frac{a}{18}$

3. Multiply $\frac{2x+3y}{2a}$ by $\frac{2a}{5x}$. *Ans.* $\frac{2x+3y}{5x}$.

4. Multiply $\frac{a^2-x^2}{2y}$ by $\frac{2a}{a+x}$. *Ans.* $\frac{(a-x)a}{y}$

5. Multiply[*] $\frac{x^2-y^2}{x}$, $\frac{x}{x+y}$ and $\frac{a}{x-y}$. *Ans.* a.

6. Multiply $3a$, $\frac{x+1}{2a}$ and $\frac{x-1}{a+b}$ together. *Ans.* $\frac{3(x^2-1)}{2(a+b)}$.

N. B. Reduce mixed quantities to improper fractions.

7. What is the continued product of $\frac{a^2-x^2}{a+b}$, $\frac{a^2-b^2}{ax+x^2}$ and $a+\frac{ax}{a-x}$? *Ans.* $\frac{a^2(a-b)}{x}$.

8. Multiply $\frac{4y^2}{5y-10}$ by $\frac{15y-30}{2y}$. *Ans.* $6y$.

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[*] Separate into factors when separation is obvious.

9. Multiply $\frac{a^4-b^4}{a+b}$ by $\frac{a^2}{ab-b^2}$. *Ans.* $\frac{a^2(a^2+b^2)}{b}$.

10. Multiply $\frac{a^2x-x^3}{a}$, by $\frac{6a}{2ax-2x^2}$. *Ans.* $3(a+x)$.

11 Required the continued product of $\frac{a^4-x^4}{a^2-b^2}$, $\frac{a+b}{a^2+x^2}$ and $\frac{a-b}{a-x}$. *Ans.* $(a+x)$.

12. Multiply $a+\frac{x}{b}$ by $a-\frac{y}{b}$. *Ans.* $\frac{a^2b^2+abx-aby-xy}{b^2}$

13. Multiply $\frac{x^2-b^2}{bc}$ by $\frac{x^2+b^2}{b+c}\times\frac{bc}{x-b}$.

Ans. $\frac{(x+b)(x^2+b^2)}{b+c}$

14. Multiply $\frac{3x^2-5x}{14}$ by $\frac{7a}{2x^2-3x}$. *Ans.* $\frac{3ax-5a}{4x^2-6}$.

15. Multiply $\frac{4a^2-16b^2}{a-2b}$ by $\frac{5b}{8a^2+32ab+32b^2}$ *Ans.* $\frac{5b}{2a+4b}$.

CASE 8 *Division of Fractions.*

(Art. 34.) To acquire a clear understanding of division in fractions, let us return to division in whole numbers.

The first principle to which we wish to call the attention of the reader, is, that if we multiply or divide both dividend and divisor of any sum in division, by any number whatever, we do not affect or change the quotient. (Art. 21.)

Thus, 2)6(3 4)12(3 8)24(3 &c

The second principle to which we would call observation is, that if we multiply any fraction by its denominator, we have the numerator for a product.

Thus, $\frac{1}{3}$ multiplied by 3 gives 1, the numerator, and $\frac{2}{5}$ by 5 gives 2, and $\frac{a}{b}$ multiplied by b gives a, &c.

Now let it be required to divide $\frac{a}{b}$ by $\frac{c}{d}$.

The quotient will be the same if we multiply both dividend and divisor by the same quantity. Let us multiply both terms by d, the denominator of the divisor, and we have $\frac{ad}{b}$ to be divided by the whole number c. But to divide a fraction by a whole number we must multiply the denominator by that number. (Art. 33.) Hence $\frac{ad}{bc}$ is the true quotient required.

We can mechanically arrive at the same result by inverting the terms of the divisor, and then multiplying the upper terms together for a numerator, and the lower terms for a denominator; therefore to divide one fraction by another we have the following

Rule. *Invert the terms of the divisor, and proceed as in multiplication.*

EXAMPLES.

1. Divide $\frac{a+b}{c}$ by $\frac{c}{a+b}$. *Ans.* $\frac{(a+b)^2}{c^2}$.

2. Divide $\frac{5x}{a}$ by $\frac{b}{c}$.

Operation: divisor inverted $\frac{c}{b}\times\frac{5x}{a}=\frac{5cx}{ab}$ *Ans.*

3. Divide $\frac{15ab}{a-x}$ by $\frac{10ac}{a^2-x^2}$.*

Operation, $\frac{15ab}{a-x}\times\frac{(a+x)(a-x)}{10ac}$. *Ans.* $\frac{3b(a+x)}{2c}$.

4. Divide $\frac{2x^2-7}{x+a}$ by $\frac{a^2}{x^2+2ax+a^2}$. *Ans.* $\frac{(2x^2-7)(x+a)}{a^2}$.

5. Divide $\frac{x^4-b^4}{x^2-2bx+b^2}$ by $\frac{x+b}{x-b}$. *Ans.* x^2+b^2.

Operation, $\frac{(x^2-b^2)(x^2+b^2)}{(x-b)(x-b)}\times\frac{x-b}{x+b}=x^2+b^2$.

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* Divide into factors, in all such cases, and cancel.

6. Divide $\frac{2ax+x^2}{a^3-x^3}$ by $\frac{x}{a-x}$. *Ans.* $\frac{2a+x}{a^2+ax+x^2}$.

7. Divide $\frac{14x-3}{5}$ by $\frac{10x-4}{25}$. *Ans* $\frac{70x-15}{10x-4}$.

8. Divide $\frac{9x^2-3x}{5}$ by $\frac{x^2}{5}$. *Ans.* $\frac{9x-3}{x}$.

9. Divide $\frac{6x-7}{x+1}$ by $\frac{x-1}{3}$. *Ans.* $\frac{18x-21}{x^2-1}$.

10. Divide $\frac{x+x^2}{3a^2}$ by $\frac{2ax+2ax^2}{7}$. *Ans.* $\frac{7}{6a^3}$

11. Divide $\frac{a^6-x^6}{a^2-2ax+x^2}$ by $\frac{a^2+ax+x^2}{a-x}$. *Ans.* a^3+x^3

12. Divide $\frac{9y^2-3y}{5}$ by $\frac{y^2}{5}$. *Ans.* $\frac{9y-3}{y}$

13. Divide $\frac{na-nx}{a+b}$ by $\frac{ma-mx}{a+b}$. *Ans.* $\frac{n}{m}$

14. Divide 12 by $\frac{(a+x)^2}{x}-a$. *Ans.* $\frac{12x}{a^2+ax+x^2}$

15. Divide $\frac{ab+bx}{x}$ by $\frac{a}{x}$. *Ans.* $b+\frac{bx}{a}$

16. Divide $\frac{x-b}{6c^2x}$ by $\frac{3cx}{4d}$. *Ans.* $\frac{4d(x-b)}{18c^3x^2}$

17. Divide a by the product of $\frac{x}{x+y}$ into $\frac{a}{x-y}$.

Ans. $\frac{x^2-y^2}{x}$.

18. Divide $\frac{3(x^2-1)}{2(a+b)}$ by the product of $\frac{(x+1)}{2a}$ into $\frac{x-1}{a+b}$.

Ans. $3a$.

SECTION II.

CHAPTER I.

Preparatory to the solution of problems, and to extended investigations of scientific truth, we commenced by explaining the reason and the manner of adding, subtracting, multiplying, and dividing algebraic quantities, both whole and fractional, that the mind of the pupil need not be called away to the art of performing these operations, when all his attention may be required on the nature and philosophy of the problem itself.

For this reason we did not commence with problems.

Analytical investigations are mostly carried on by means

OF EQUATIONS.

(Art. 35.) An equation is an algebraical expression, meaning that certain quantities are equal to certain other quantities. Thus, $3+4=7$; $a+b=c$; $x+4=10$, are equations, and express that 3 added to 4 is equal to 7, and in the second equation that a added to b is equal to c, &c. The signs are only abbreviations for words.

The quantities on each side of the sign of equality are called *members*. Those on the left of the sign form the *first* member, those on the right the *second.*

In the solution of problems every equation is supposed to contain at least one *unknown quantity*, and the solution of an equation is the art of changing and operating on the terms by means of addition, subtraction, multiplication, or division, or by all these combined, so that the unknown term may stand alone as one member of the equation, equal to known terms in the other member, by which it then becomes known.

Equations are of the first, second, third, or fourth degree, according as the unknown quantity which they contain is of the first, second, third, or fourth power.

$ax+b=3ax$ is an equation of the *first degree* or *simple equation.*

$ax^2+bx=3ab$ is an equation of the *second degree* or *quadratic equation.*

$ax^3+bx^2+cx=2a^4b$ is an equation of the *third degree.*

$ax^4+bx^3+cx^2+dx=2ab^5$ is an equation of the *fourth degree.*

We shall at present confine ourselves to simple equations.

(Art. 36.) The unknown quantity of an equation may be united to known quantities, in *four* different ways: by addition, by subtraction, by multiplication, and by division, and further by various combinations of *these four* ways as shown by the following equations, both numeral and literal:

	NUMERAL.	LITERAL.
1st. By addition,	$x+6=10$	$x+a=b$
2d. By subtraction,	$x-8=12$	$x-c=d$
3d. By multiplication,	$20x=80$	$ax=e$
4th. By division,	$\frac{x}{4}=16$	$\frac{x}{d}=g+a.$

5th. $x+6-8+4=10+2-3$, $x+a-b+c=d+c$, &c., are equations in which the *unknown* is connected with known quantities by both addition and subtraction.

$2x+\frac{x}{3}=21$, $ax+\frac{x}{b}=c$, are equations in which the *unknown* is connected with known quantities by both multiplication and division.

Equations often occur, in solving problems, in which all of these operations are combined.

(Art. 37.) Let us now examine how the *unknown* quantity can be separated from others, and be made to stand by itself Take the 1st equation, or other similar ones.

$x+6=10$ $x+a=b$

Take equal quantities $6=6$ $a=a$ from both

members, and $x=10-6$ $x=b-a$ the

remainders must be equal. (Ax. 2.) Now we find the term added to x, whatever it may be, appears on the other side with a contrary sign, and the unknown term x being equal to known terms is now known.

Take the equations	$x-8=12$	$x-c=d$	
Add equals to both memb.	$8=8$	$c=c$	
Sums are equal	$x=12+8$	$x=d+c$	(Ax. 1.)

Here again the quantity united to x appears on the opposite side with a contrary sign.

From this we may draw the following principle or rule of operation:

Any term may be transposed from one member of an equation to the other, by changing its sign.

Now $20x=80$. $ax=e$. If we divide both members by the coefficient of the unknown term, the quotients will be equal. (Ax. 4.) Hence $x=\frac{80}{20}=4$. $x=\frac{e}{a}$.

That is, the unknown quantity is disengaged from known quantities, in this case, by *division.*

Again, take the equations $\frac{x}{4}=16$; $\frac{x}{d}=g+a$.

Multiply both members by the divisor of the unknown term, and we have $x=16\times4$. $x=gd+ad$. Equations which must be true by (Ax. 3.), and here it will be observed that x is liberated by *multiplication.*

From these observations we deduce this general principle:

That to separate the unknown quantity from additional terms we must use subtraction; from subtracted terms we must use addition; from multiplied terms we must use division; from divisors we must use multiplication.

In all cases take the opposite operation.

EXAMPLES.

1. Given $3x-4=7x-16$ to find the value of x. *Ans.* $x=3$.

2. Given $3x+9-1-5x=0$ to find the value of x.
Ans. $x=4$.

3. Given $4y+7=y+21-3+y$ to find y. *Ans.* $y=5\frac{1}{2}$.

4. Given $5ax-c=b-3ax$ to find the value of x.
Ans. $x=\frac{b+c}{8a}$

5. Given $ax^2+bx=9x^2+cx$ to find the value of x in terms of a, b, and c. *Ans.* $x=\frac{c-b}{a-9}$.

N. B. In this last example we observe that every term of the equation contains at least one factor of x; we therefore divide every term by x, to suppress this factor.

(Art. 38.) In many problems, the unknown quantity is often combined with known quantities, not merely in a simple manner, but under various fractional and compound forms.—Hence, rules can only embody general principles, and skill and tact must be acquired by close attention and practical application: but from the foregoing principles we draw the following

General Rule. *Connect and unite as much as possible all the terms of a similar kind on both sides of the equation. Then, to clear of fractions, multiply both sides by the denominators, one after another, in succession. Or, multiply by their continued product, or by their least common multiple,* (when such a number is obvious,) *and the equation will be free of fractions.*

Then, transpose the unknown terms to the first member of the equation, and the known terms to the other. Then unite the similar terms, and divide by the coefficient of the unknown term, and the equation is solved.

EXAMPLES.

1. Given $x+\frac{1}{2}x+3-7=6-1$, to find the value of x. Uniting the known terms, after transposition, agreeably to the rule of addition, we find $x+\frac{1}{2}x=9$. Multiply every term by 2, and we have $2x+x=18$. Therefore $x=6$.

2. Given $2x+\frac{3}{4}x+\frac{1}{3}x-3a=4b+3a$, to find x.

N. B. We may clear of fractions, in the first place, before we condense and unite terms, if more convenient, and among literal quantities this is generally preferable.

In the present case let us multiply every term of the equation by 12, the product of 3×4, and we shall have

$$24x+9x+4x-36a=48b+36a.$$

Transpose and unite, and $37x=48b+72a$

Divide by 37, and $x=\frac{48b+72a}{37}$.

3. Given $\frac{1}{2}x+\frac{1}{3}x+\frac{1}{4}x=39$, to find the value of x.

Here are no scattering terms to collect, and clearing of fractions is the first operation.

By an examination of the denominators, 12 is obviously their least common multiple, therefore multiply by 12. Say 12 halves are 6 whole ones, 12 thirds are 4, 12 fourths are 3, &c.

Hence, $6x+4x+3x=39\times12$

Collect the terms, $13x=39\times12$

Divide by 13, and $x=3\times12=36$, *Ans.*

N. B. In other books we find the numerals actually multiplied by 12. Here it is only indicated, which is all that is necessary. For when we come to divide by the coefficient of x, we shall find factors that will cancel, unless that coefficient is prime to all the other numbers used, which, in practice, is very rarely the case.

4. Given $\frac{1}{2}x+\frac{1}{3}x+\frac{1}{4}x=a$, to find x.

This example is essentially the same as the last. It is identical if we suppose $a=39$.

Solution, $6x+4x+3x=12a$

Or, $13x=12a$

Divide and $x=\frac{12a}{13}$

Now if a be any multiple of 13, the problem is easy and brief in numerals.

5. Given $21+\frac{3x-11}{16}=\frac{5x-5}{8}+\frac{97-7x}{2}$ to find the value of x.

Here 16 is obviously the least common multiple of the denominators, and the rule would require us to multiply by it, and such an operation would be correct; but in this case it is more easy to multiply by the least denominator 2, and then condense like terms. Thus, Multiply by 2, and we have

$$42+\frac{3x-11}{8}=\frac{5x-5}{4}+97-7x.$$

Recollect that we can multiply a fraction by dividing its denominator. Also observe that we can mentally take away 42 from both sides of the equation, and the remainders will be equal (Ax. 2.)

Then $$\frac{3x-11}{8}=\frac{5x-5}{4}+55-7x,$$

Multiply by 8, and

$$3x-11=10x-10+440-56x\,;$$

Transposing and uniting terms, we have

$$49x=441\,;$$

By division, $$x=9.$$

6. Given $\frac{3}{5}x+2\frac{1}{2}+11=\frac{2}{8}x+17$, to find x.

If we commence by clearing of fractions, we should make comparatively a long and tedious operation. Let us first reduce it by striking out equals from both sides of the equation. *We can take* 11 *from both sides* without any formality of transposing or changing signs; say drop equals from both sides, (Ax. 2.) and reduce the fraction $\frac{2}{8}x=\frac{1}{4}x$.

All this can be done as quick as thought, and we shall have

$$\frac{3}{5}x+2\frac{1}{2}=\frac{1}{4}x+6\,;$$

Multiply by 4, then

$$\frac{12x}{5}+10=x+24,\quad\text{or}\quad\frac{12x}{5}=x+14\,;$$

Hence, $7x=70$, or $x=10$ *Ans.*

7. Given $\frac{1}{3}x-5+\frac{1}{4}x+8+\frac{1}{5}x-10=100-6-7$ to find the value of x.

Collecting and uniting the numeral quantities, we have

$$\frac{1}{3}x+\frac{1}{4}x+\frac{1}{5}x=94\,;$$

Multiply every term by 60, and we have

$$20x+15x+12x=94.60$$

Collecting terms, $$47x=94.60$$

Divide both sides by 47, and $x=2.60=120$ *Ans.*

(Art. 30.) When equations contain *compound* fractions and simple ones, clear them of the simple fractions first, and unite, as far as possible, all the simple terms.

EXAMPLES.

8. Given $\frac{6x+7}{9}+\frac{7x-13}{6x+3}=\frac{2x+4}{3}$ to find the value of x.

Multiply all the terms by the smallest denominator, 3. That is, divide all the *denominators* by 3, and

$$\frac{6x+7}{3}+\frac{7x-13}{2x+1}=2x+4.$$

Multiply by 3 again, and $6x+7+\frac{21x-39}{2x+1}=6x+12$

Drop $6x+7$, and $\frac{21x-39}{2x+1}=5.$

Clear of fractions, $21x-39=10x+5.$

Drop $10x$ and add 39, and we have $11x=44$, or $x=4.$

9. Given $\frac{7x+16}{21}-\frac{x+8}{4x-11}=\frac{x}{3}$ to find the value of x.

Observe that $\frac{7x+16}{21}$ may be expressed in two parts, thus, $\frac{7x}{21}+\frac{16}{21}$. Observe also, that $\frac{7x}{21}=\frac{x}{3}$. Hence these terms may be dropped, the remainders must be *equal*. Transpose the minus term, then $\frac{16}{21}=\frac{x+8}{4x-11}$.

Clear of fractions, and $64x-11\times16=21x+21\times8.$

Drop $21x$ and observe that 11×16 is the same as $22\times8.$

Then $43x-22\times8=21\times8.$ Let $a=8,$

Then $43x-22a=21a.$

Transpose $-22a$ and $43x=43a.$

Hence $x=a.$ But $a=8.$ Therefore $x=8.$

N. B. We operate thus, to call attention to the relation of quantities, and to form a habit of quick comparison, which will, in many instances, save much labor and introduce the pupil into the true spirit of the science

10. Given $\frac{9x+20}{36}=\frac{4x-12}{5x-4}+\frac{x}{4}$ to find the value of x.

By a slight examination we perceive that $\frac{9x}{36}$ is equal to $\frac{1}{4}x$. Hence these terms may be left out, as they balance each other. Also, $\frac{20}{36}=\frac{5}{9}$.

$$\text{Therefore } \frac{5}{9}=\frac{4x-12}{5x-4}.$$

Clear of fractions, and $25x-20=36x-108$.
Transpose $25x-36x=20-108$.
Unite and change signs, and $11x=88$ or $x=8$, *Ans*

11. Given $\frac{20x}{25}+\frac{36}{25}+\frac{5x+20}{9x-16}=\frac{4x}{5}+\frac{86}{25}$ to find x.

By taking equals from both sides, we have

$$\frac{5x+20}{9x-16}=2. \quad \text{By reduction } x=4.$$

12. Given $\frac{3x}{4}-\frac{x-1}{2}=6x-\frac{20x+13}{4}$ to find x.

Multiply by 4, to clear of fractions, and

$$3x-2x+2=24x-20x-13. \quad \text{Reduced } x=5.$$

(Art. 40.) When a *minus* sign stands before a compound quantity, it indicates that the whole is to be subtracted; but we subtract by changing signs, (Art. 5). The minus sign before $\frac{x-1}{2}$ in the last example, does not indicate that the x is minus, but that this term must be subtracted. When the term is multiplied by 4, the numerator becomes $2x-2$, and subtracting it we have $-2x+2$.

Having thus far explained, we give the following unwrought equations, for practice:

13. Given $\frac{3x}{2}=\frac{x}{4}+24$ to find the value of x. *Ans.* $19\frac{1}{5}$.

14. Given $\frac{1}{4}x+\frac{1}{6}x=10$ to find the value of x. *Ans.* 24.

15. Given $\frac{x-3}{2}+\frac{x}{3}=20-\frac{x+19}{2}$ to find x. *Ans* 9

16. Given $\frac{x+1}{2}+\frac{x+2}{3}=16-\frac{x+3}{4}$ to find x. *Ans.* 13

17. Given $2x-\frac{x+3}{3}+15=\frac{12x+26}{5}$ to find x.

Ans. $x=12$.

18. Given $x-\frac{2x+1}{3}=\frac{x+3}{4}$ to find x. *Ans.* $x=13$.

19. Given $\frac{1}{2}x+\frac{1}{3}x+\frac{1}{4}x+\frac{1}{5}x=77$ to find x. *Ans.* $x=60$.

20. Given $\frac{1}{2}x+\frac{1}{3}x+\frac{1}{4}x=130$ to find x. *Ans.* $x=120$

21. Given $\frac{1}{2}x+\frac{1}{6}x+\frac{1}{12}x=90$ to find x. *Ans.* $x=120$

22. Given $\frac{1}{2}y+\frac{1}{3}y+\frac{1}{7}y=82$ to find y. *Ans.* $y=84$

23. Given $5x+\frac{1}{6}x+\frac{1}{2}x=34$ to find x. *Ans.* $x=6$.

24. Given $11x+\frac{2x}{3}+\frac{5x}{4}+\frac{x}{6}+\frac{x}{24}=315$ to find x.

Ans. $x=24$.

25. Given $y+\frac{y}{2}+\frac{3y}{4}+\frac{2y}{7}+\frac{y}{14}=146$ to find y. *Ans.* $y=56$

26. Given $\frac{5x+5}{x+2}-29=\frac{6x-12}{x-2}-30$ to find x. *Ans.* $x=2$

There is a peculiar circumstance attending this 26th example, and the 4th example of Art. 42, which will cause us to refer to them in a subsequent part of this work.

N. B. In solving equations 19, 20, 21, 22, and 23, use no larger numbers than those given, indicating and not performing numeral multiplications.

(Art. 41.) Every proportion may be converted into an equation. Proportion is nothing more than an assumption that the same *relation* or the same *ratio* exists between two quantities, as exists between two other quantities.

That is, A is to B as C is to D. There is *some relation* between A and B. Let r express that relation, then $B=rA$. But the relation between C and D is the same (by hypothesis) as between A and B. Hence $D=rC$.

$$\text{Then in place of } A : B :: C : D$$
$$\text{we have } A : rA :: C : rC$$

Multiply the extreme terms, and we have rCA.

Multiply the mean terms, and we have rAC.

Obviously the same product, whatever quantities may be represented by either A, or r, or C.

Hence, to convert a proportion into an equation, we have the following

Rule. *Place the product of the extremes equal to the product of the means.*

(Art. 42.) The relation between two quantities is not changed by multiplying or dividing both of them by the same quantity. Thus, $a : b :: 2a : 2b$, or more generally, $a : b :: na : nb$, for the product of the extremes is obviously equal to the product of the means.

That is, a is to b as any number of times a is to the same number of times b.

We shall take up proportion again, but Articles 41 and 42 are sufficient for our present purpose.

EXAMPLES.

1. Given $3x-1 : 2x+1 :: 3x : x$ to find x.

(By Art. 41.) $3x^2-x=6x^2+3x$.

Transpose and unite, and we have $0=3x^2+4x$.

Divide by x, and $3x+4=0$ or $x=-\frac{4}{3}$, *Ans.*

2. Given $\frac{3}{2} : \frac{3x}{4} :: 6 : 5x-4$ to find x.

The first two terms have the same relation as $\frac{1}{2} : \frac{1}{4}x$, or as $2 : x$. Hence $2 : x :: 6 : 5x-4$.

Product of extremes and means, $10x-8=6x$ or $x=2$.

3. Given $\frac{(x-1)(x+1)}{3a} : \frac{x+1}{3a} :: 2 : 1$ to find x.

Ans. $x=3$.

4. Given $\frac{x-5}{4}:(x-5)::\frac{2}{3}:\frac{3}{4}$ to find x.

Ans. $x=5$.

5. Given $x+2:a::b:c$ to find the value of x.

Ans. $x=\frac{ab}{c}-2$.

6. Given $2x-3:x-1::2x:x+1$ to find the value of x.

Ans. $x=3$

7. Given $x+6:38-x::9:2$ to find x. *Ans.* $x=30$.

8. Given $x+4:x-11::100:40$ to find x. *Ans.* $x=21$.

QUESTIONS PRODUCING SIMPLE EQUATIONS.

(Art. 43.) We now suppose the pupil can readily reduce a simple equation containing but one unknown quantity, and he is, therefore, prepared to solve the following questions. The only difficulty he can experience is the want of tact to reason briefly and powerfully with algebraic symbols; but this tact can only be acquired by practice and strict attention to the solution of questions. We can only give the following general direction:

Represent the unknown quantity by some symbol or letter, and really consider it as definite and known, and go over the same operations as to verify the answer when known.

EXAMPLES.

1. What number is that whose third part added to its fourth part makes 21? *Ans.* 36.

The number may be represented by x.

Then $\frac{1}{3}x+\frac{1}{4}x=21$. Therefore $x=36$.

2. Two men having found a bag of money, disputed about the division of it. One said that the half, the third, and the fourth parts of it made \$130, and if the other could tell how much money the bag contained, he might have it all. How much money did the bag contain? *Ans.* \$120.

(See equation 20, Art. 40.)

3. A man has a lease for 20 years, one-third of the time pas. is equal to one-half of the time to come. How much of the timu nas passed?

Let $x=$ the time past.
Then $20-x=$ the time to come.

By the question $\frac{x}{3}=\frac{20-x}{2}$. Therefore $x=12$ *Ans.*

4. What number is that, from which 6 being subtracted, and the remainder multiplied by 11, the product will be 121?

Let $x=$ the number.
Then $(x-6)11=121$, or $x-6=11$ by division.
Hence $x=17$.

5. It is required to find two numbers, whose difference is 6, and if $\frac{1}{3}$ of the less be added to $\frac{1}{5}$ of the greater, the sum will be equal to $\frac{1}{3}$ of the greater diminished by $\frac{1}{5}$ of the less.

Let $x=$ the less. Then $x+6=$ the greater.

By the question $\frac{1}{3}x+\frac{x+6}{5}=\frac{x+6}{3}-\frac{1}{5}x$.

Drop $\frac{1}{3}x$ from both sides and add $\frac{1}{5}x$ to both sides, and we have $\frac{2x+6}{5}=2$, or $x=2$, the less number.

We may clear of fractions in full, and then transpose and unite terms, but the operation would be much longer.

6. After paying $\frac{1}{4}$ and $\frac{1}{5}$ of my money, I had \$66 left; how much had I at first? *Ans.* \$120

7. After paying away $\frac{1}{3}$ of my money, and $\frac{1}{4}$ of what remained, and losing $\frac{1}{5}$ of what was left, I found that I had still \$24. How much had I at first? *Ans.* 60.

8. What number is that from which if 5 be subtracted, $\frac{2}{3}$ of the remainder will be 40? *Ans.* 65.

9. A man sold a horse and a chaise for \$200; $\frac{1}{2}$ of the price of the horse was equal to $\frac{1}{3}$ of the price of the chaise. What was the price of each? *Ans.* Chaise \$120. Horse \$80

10 Divide 48 into two such parts, that if the less be divided by 4, and the greater by 6, the sum of the quotients will be 9.

Ans. 12 and 36.

11. An estate is to be divided among 4 children, in the following manner:

The first is to have \$200 more than $\frac{1}{4}$ of the whole.
The second is to have \$340 more than $\frac{1}{5}$ of the whole.
The third is to have \$300 more than $\frac{1}{6}$ of the whole.
And the fourth is to have \$400 more than $\frac{1}{8}$ of the whole.
What is the value of the estate? *Ans.* \$4800

12. Find two numbers in the proportion of 3 to 4, whose sum shall be to the sum of their squares as 7 to 50.

Ans. 6 and 8.

N. B. When proportional numbers are required, it is generally most convenient to represent them by one unknown term, with coefficients of the given relation. Thus, numbers in proportion of 3 to 4, may be expressed by $3x$ and $4x$, and the proportion of a to b may be expressed by ax and bx.

13. The sum of \$2000 was bequeathed to two persons, so that the share of *A* should be to that of *B* as 7 to 9. What was the share of each? *Ans.* *A*'s share \$875, *B*'s share \$1125.

14. A certain sum of money was put at simple interest, and in 8 months it amounted to \$1488, and in 15 months it amounted to \$1530. What was the sum? *Ans.* \$1440.

Let $x=$ the sum. The sum or principal subtracted from the amount will give interest: therefore $1488-x$ represents the interest for 8 months, and $1530-x$ is the interest for 15 months.

Now whatever be the rate per cent. double time will give double interest, &c. Hence $8 : 15 :: 1488-x : 1530-x$.

N. B. To acquire true delicacy in algebraical operations, it is often expedient not to use large numerals, but let them be represented by letters. In the present example let $a=1488$ Then $a+42=1530$, and the proportion becomes $8 : 15 :: a-x : a+42-x$.

Multiply extremes, &c., $8a+8\cdot42-8x=15a-15x$.

Drop $8a$ and $-8x$. We then have $8\cdot42=7a-7x$.

Dividing by 7 and transposing $x=a-48=1440$, *Ans.*

15. A merchant allows \$1000 per annum for the expenses of his family, and annually increases that part of his capital which is not so expended by a third of it; at the end of three years his original stock will double. What had he at first?

Ans. \$14,800.

Let $x=$ the original stock, and $a=1000$.

To increase any quantity by its $\frac{1}{3}$ part is equivalent to multiplying it by $\frac{4}{3}$. Hence $\frac{4x-4a}{3}$ is his 2d year's stock.

16. A man has a lease for 99 years, and being asked how much of it had already expired, answered that $\frac{2}{3}$ of the time past was equal to $\frac{4}{5}$ of the time to come. Required the time past and the time to come.

Assume $a=99$. *Ans.* Time past, 54 years.

17. In the composition of a quantity of gunpowder
The nitre was 10 lbs. more than $\frac{2}{3}$ of the whole,
The sulphur $4\frac{1}{2}$ lbs. less than $\frac{1}{6}$ of the whole,
The charcoal 2 lbs. less than $\frac{1}{7}$ of the nitre.
What was the amount of gunpowder? *Ans.* 69 lbs

18. Divide \$183 between two men, so that $\frac{4}{7}$ of what the first receives shall be equal to $\frac{3}{10}$ of what the second receives. What will be the share of each? *Ans.* 1st, \$63; 2d, \$120.

19. Divide the number 68 into two such parts that the difference between the greater and 84 shall be equal to 3 times the difference between the less and 40. *Ans.* Greater, 42; Less, 26.

20. Four places are situated in the order of the letters *A*, *B*, *C*, *D* The distance from *A* to *D* is 34 miles. The distance from *A* to *B* is to the distance from *C* to *D* as 2 to 3. And $\frac{1}{4}$ of the distance from *A* to *B*, added to half the distance from *C* to *D*, is three times the distance from *B* to *C*. What are the respective distances?

Ans. From *A* to $B=12$; from *B* to $C=4$; from *C* to $D=18$.

21. A man driving a flock of sheep to market, was met by a party of soldiers, who plundered him of $\frac{1}{2}$ of his flock and 6 more. Afterwards he was met by another company, who took $\frac{1}{2}$ what he then had and 10 more: after that he had but 2 left. How many had he at first? *Ans.* 45.

22. A laborer engaged to serve for 60 days on these conditions: That for every day he worked he should have 75 cents and his board, and for every day he was idle he should forfeit 25 cents for damage and board. At the end of the time a settlement was made and he received $25. How many days did he work, and how many days was he idle?

The common way of solving such questions is to let $x=$ the days he worked; then $60-x$ represents the days he was idle. Then sum up the account and put it equal to $25.

Another method is to consider that if he worked the whole 60 days, at 75 cents per day, he must receive $45. But for every day he was idle, he not only lost his wages, 75 cents, but 25 cents in addition. That is, he lost $1 every day he was idle.

Now let $x=$ the days he was idle. Then $x=$ the dollars he lost. And $45-x=25$ or $x=20$ the days he was idle.

23. A boy engaged to carry 100 glass vessels to a certain place, and to receive 3 cents for every one he delivered, and to forfeit 9 cents for every one he broke. On settlement, he received 2 dollars and 40 cents. How many did he break?

Ans. 5.

24. A person engaged to work a days on these conditions: For each day he worked he was to receive b cents, for each day he was idle he was to forfeit c cents. At the end of a days he received d cents. How many days was he idle?

Ans. $\frac{ab-d}{b+c}$ days.

25. It is required to divide the number 204 into two such parts, that $\frac{2}{5}$ of the less being taken from the greater, the remainder will be equal to $\frac{3}{7}$ of the greater subtracted from 4 times the less. *Ans.* The numbers are 154 and 50.*

(Art. 44.) We introduce this, and a few following problems, to teach one important *expedient*, not to say principle, which is, not always to commence a problem by putting the unknown quantity equal to a *single letter*. We may take $2x$, $3x$, or nx to represent the unknown quantity, as well as x, and we may resort to this expedient when *fractional parts* of the quantity are called in question, and take such a number of x's as may be divided without fractions.

In the present example we do not put $x=$ to the less part, as we must have $\frac{2}{5}$ of the less part. It will be more convenient to put $5x=$ the less part. Then $\frac{2}{5}$ of it will be $2x$. Put $a=204$.

26. A man bought a horse and chaise for 341 (a) dollars. Now if $\frac{3}{8}$ of the price of the horse be subtracted from twice the price of the chaise, the remainder will be the same as if $\frac{5}{7}$ of the price of the chaise be subtracted from 3 times the price of the horse. Required the price of each.

Ans. Horse \$152. Chaise \$189.

N. B. Let $8x=$ the price of the horse.
Or let $7x=$ the price of the chaise.

Solve this question by both of these notations.

27. From two casks of equal size are drawn quantities, which are in the proportion of 6 to 7; and it appears that if 16 gallons less had been drawn from that which now contains the less, only one half as much would have been drawn from it as from the other. How many gallons were drawn from each? *Ans.* 24 and 28.

N. B. Let $6x$ and $7x$ equal the quantities drawn out.

28. Divide \$315 among four persons, *A*, *B*, *C*, and *D*, giving *B* as much and $\frac{1}{2}$ more than *A*; *C* $\frac{1}{3}$ more than *A* and *B* together; and *D* $\frac{1}{4}$ more than *A*, *B* and *C*. What is the share of each? *Ans.* *A* \$24. *B* \$36. *C* \$80, and *D* \$175.

If we take x to represent *A*'s share, we shall have a very complex and troublesome problem.* But it will be more simple by making $6x=$*A*'s share.

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* Taking x for *A*'s share, and reducing their sum, gives Equation 24, Art. 40.

Thus, let $6x=A$'s share.
Then $9x=B$'s share.
And $15x+5x=C$'s share.
Also $35x+\frac{35x}{4}=D$'s share.

Sum $70x+\frac{35x}{4}=315$

$$280x+35x=315\times 4$$

$$315x=315\times 4$$

$x=4$ Hence $6x=24$, A's sh

29. A gamester at play staked $\frac{1}{5}$ of his money, which he lost, but afterwards won 4 shillings; he then lost $\frac{1}{4}$ of what he had, and afterwards won 3 shillings; after this he lost $\frac{1}{6}$ of what he had, and finding that he had but 20 shillings remaining, he left off playing. How much had he at first? *Ans.* 30 shillings.

30. A gentleman spends $\frac{2}{3}$ of his yearly income for the support of his family, and $\frac{2}{3}$ of the remainder for improving his house and grounds, and lays by \$70 a year. What is his income? *Ans.* 9×70 dollars, or more generally, 9 times the sum he saves.

31. Divide the number 60 (a) into two such parts that their product may be equal to three times the square of the less number? *Ans.* 15 and 45, or $\frac{1}{4}a=$ the less part.

32. After paying away $\frac{1}{4}$ and $\frac{1}{7}$ of my money, I had 34 (a) dollars left. What had I at first?

Ans. 56 dollars. General answer $\frac{a}{17}\times 28$.

33. My horse and saddle are together worth 90 (a) dollars, and my horse is worth 8 times my saddle. What is the value of each? *Ans.* Saddle \$10. Horse \$80.

34. My horse and saddle are together worth a dollars, and my horse is worth n times my saddle. What is the value of each? *Ans.* Saddle $\frac{a}{n+1}$. Horse $\frac{na}{n+1}$.

35. The rent of an estate is 8 per cent greater this year than last. This year it is 1890 dollars. What was it last year?

Ans. $1750

36 The rent of an estate is n per cent. greater this year than last. This year it is a dollars. What was it last year?

Ans. $\frac{100a}{100+n}$ dollars.

37 *A* and *B* have the same income. *A* contracts an annual debt amounting to $\frac{1}{7}$ of it; *B* lives upon $\frac{4}{5}$ of it; at the end of two years *B* lends to *A* enough to pay off his debts, and has 32 (a) dollars to spare. What is the income of each?

Ans. $280 or $\frac{1}{4}(35a)$.

38. What number is that of which $\frac{1}{3}$, $\frac{1}{4}$ and $\frac{2}{7}$ added together make 73 (a)?

Ans. 84. General *Ans.* $\frac{84a}{73}$.

39. *A* person after spending 100 dollars more than $\frac{1}{5}$ of his income, had remaining 35 dollars more than $\frac{1}{2}$ of it. Required his income. *Ans.* $450.

40. A person after spending (a) dollars more than $\frac{1}{7}$ of his income, had remaining (b) dollars more than $\frac{1}{3}$ of it. Required his income.

Ans. $\frac{21(a+b)}{11}$ dollars.

41. There are two numbers in proportion of 2 to 3, and if 4 be added to each of them, the sums will be in proportion of 5 to 7? *Ans.* 16 and 24.

42. It is required to find a number such, that if it be increased by 7, the square root of the sum shall be equal to the square root of the number itself, and 1 more. *Ans.* 9.

43. *A* sets out from a certain place, and travels at the rate of 7 miles in 5 hours; and 8 hours afterward *B* sets out from the same place in pursuit, at the rate of 5 miles in 3 hours. How long and how far must *B* travel before he overtakes *A*?

Ans. 42 hours, and 70 miles

SIMPLE EQUATIONS.

CHAPTER II.

(Art. 45.) We have given a sufficient number of examples, and introduced the reader sufficiently far into the science previous to giving instructions for the solution of questions containing two or more unknown quantities.

There are many simple problems which one may meet with in algebra which cannot be solved by the use of a single *unknown* quantity, and there are also some which *may be solved* by a single unknown letter, that may become much more simple by using two or more unknown quantities.

When two unknown quantities are used, two independent equations must exist, in which the value of the unknown letters must be the same in each. When three unknown quantities are used, there must exist three independent equations, in which the value of any one of the unknown letters is the same in each.

In short, there must be as many independent equations as unknown quantities used in the question.

For more definite illustration let us suppose the following question:

A merchant sends me a bill of 16 *dollars for* 3 *pair of shoes and* 2 *pair of boots; afterwards he sends another bill of* 23 *dollars for* 4 *pair of shoes and* 3 *pair of boots, charging at the same rate. What was his price for a pair of shoes, and what for a pair of boots?*

This can be resolved by one unknown quantity, but it is far more simple to use two.

Let $x=$ the price of a pair of shoes,

And $y=$ the price of a pair of boots.

Then by the question $3x+2y=16$

And $4x+3y=23$.

These two equations are independent; that is, one cannot be converted into the other by multiplication or division, notwithstanding the value of x and of y is the same in both equations.

Having intimated that this problem can be resolved with one unknown quantity, we will explain in what manner, before we proceed to a general solution of equations containing two unknown quantities.

Let $x=$ the price of a pair of shoes.

Then $3x=$ the price of three pair of shoes.

And $16-3x=$ the price of two pair of boots.

Consequently $\frac{16-3x}{2}=$ the price of one pair of boots.

Now 4 pair of shoes which cost $4x$, and 3 pair of boots which cost $\frac{48-9x}{2}$ being added together, must equal 23 dollars.

That is, $4x+24-\frac{9}{2}x=23$.

Or, $1-\frac{1}{2}x=0$. Therefore $x=2$ dollars, the price of a pair of shoes. Substitute the value of x in the expression $\frac{16-3x}{2}$ and we find 5 dollars for the price of a pair of boots.

Now let us resume the equations,

$$3x+2y=16 \quad (A)$$
$$4x+3y=23 \quad (B)$$

FIRST METHOD OF ELIMINATION.

(Art. 46.) Transpose the terms containing y to the right hand sides of the equations, and divide by the coefficients of x, and

From equation (A) we have $x=\frac{16-2y}{3}$ (C)

And from (B) we have $x=\frac{23-3y}{4}$ (D)

Put the two expressions for x equal to each other. (Ax. 7.)

$$\text{And} \qquad \frac{16-2y}{3}=\frac{23-3y}{4}.$$

An equation which readily gives $y=5$, which, taken as the value of y, in either equation (C) or (D) will give $x=2$.

This method of elimination, just explained, is called th method by comparison

SECOND METHOD OF ELIMINATION.

(Art. 47.) To explain another method of solution, let us again resume the equations:

$$3x+2y=16 \qquad (A)$$
$$4x+3y=23 \qquad (B)$$

The value of x from equation (A) is $x=\frac{1}{3}(16-2y)$.

Substitute this value for x in equation (B), and we have $4\times\frac{1}{3}(16-2y)+3y=23$, an equation containing only y.

Reducing it, we find $y=5$ the same as before.

This method of elimination is called the method by substitution, and consists in finding the value of one unknown quantity from one equation to put that value in the other which will cause one unknown quantity to disappear.

THIRD METHOD OF ELIMINATION.

(Art. 48.) Resume again $3x+2y=16 \quad (A)$
$4x+3y=23 \quad (B)$

When the coefficients of either x or y are the same in both equations, and the signs alike, that term will disappear by subtraction.

When the signs are unlike, and the coefficients equal, the term will disappear by addition.

To make the coefficients of x equal, multiply each equation by the coefficient of x in the other.

To make the coefficients of y equal, multiply each equation by the coefficient of y in the other.

Multiply equation (A) by 4 and $12x+8y=64$
Multiply equation (B) by 3 and $12x+9y=69$

Difference $y=5$ as before.

To continue this investigation, let us take the equations

$$2x+3y=23 \qquad (A)$$
$$5x-2y=10 \qquad (B)$$

Multiply equation (A) by 2, and equation (B) by 3, and we have

$$4x+6y=46$$
$$15x-6y=30$$

Equations in which the coefficients of y are equal, and the signs unlike. In this case add, and the y's will destroy each other, giving $19x=76$

Or $x=4$.

This method of elimination is called the method by addition and subtraction.

FOURTH METHOD OF ELIMINATION.

(Art. 48.) Take the equations $2x+3y=23$. (A)

And $5x-2y=10$. (B)

Multiply one of the equations, for example (A), by some indeterminate quantity, say m.

$$\begin{array}{ll} \text{Then} & 2mx+3my=23m \\ \text{Subtract } (B) & \underline{5x-2y=10} \\ \text{Remainder, } (C) & (2m-5)x+(3m+2)y=23m-10 \end{array}$$

As m is an indeterminate quantity, we can assume it of any value to suit our pleasure, and whatever the assumption may be, the equation is still true.

Let us assume it of such a value as shall make the coefficient of y, $(3m+2)=0$.

The whole term will then be 0 times y, which is 0, and equation (C) becomes

$$(2m-5)x=23m-10$$

Or $$x=\frac{23m-10}{2m-5} \qquad (D)$$

But $3m+2=0$. Therefore $m=-\frac{2}{3}$.

Which substitute for m in equation (D), and we have

$$x=\frac{-23\times\frac{2}{3}-10}{-2\times\frac{2}{3}-5}=\frac{-23\times2-30}{-2\times2-15}=\frac{-76}{-19}=4.$$

This is a French method, introduced by Bezout, but it is too indirect and metaphysical to be much practised, or in fact much known.

Of the other three methods, sometimes one is preferable and sometimes another, according to the relation of the coefficients and the positions in which they stand.

No one should be prejudiced against either method, and in practice we use either one, or modifications of them, as the case may require. The forms may be disregarded when the principles are kept in view.

(Art. 49.) To present these different forms in the most general manner, let us take the following general equations, as all particular equations can be reduced to these forms.

$$ax+by=c \qquad (A)$$
$$a'x+b'y=c' \qquad (B)$$

Observe that a and a', may represent very different quantities, so b and b' may be different, also c and c' may be different. In special problems, however, a may be equal to a', or be some multiple of it; and the same remark may apply to the other letters. In such cases the solution of the equations is much easier than by the definite forms. Hence, in solving definite problems great attention should be paid to the relative values of the coefficients.

First method.

Transpose the terms containing y and divide by the coefficients of x, and

$$x=\frac{c-by}{a}, \quad \text{also} \quad x=\frac{c'-b'y}{a'} \qquad (C)$$

Therefore $$\frac{c-by}{a}=\frac{c-b'y}{a} \qquad \text{(Axiom 7.)}$$

Clearing of fractions, give $a'c-a'by=ac'-ab'y.$

Transpose, and $(ab'-a'b)y=ac'-a'c.$

By division $$y=\frac{ac'-a'c}{ab'-a'b}.$$

When y is determined, its value put in either equation marked (C) will give x.

Second method.

From equation (A) $$x=\frac{c-by}{a}.$$

Which value of x substitute in equation (B) and

$$\frac{a'c-a'by}{a}+b'y=c'.$$

Clearing of fractions and transposing $a'c$, we have

$$ab'y-a'by=ac'-a'c$$

Or $$y=\frac{ac'-a'c}{ab'-a'b}$$

The same value of y as before found.

Third method.

Multiply equation (A) by a', and equation (B) by a.

And $\quad a'ax+a'by=a'c.$

Also $\quad a'ax+ab'y=ac'$

Difference $\quad (ab'-a'b)y=ac'-a'c$

Or $$y=\frac{ac'-a'c}{ab'-a'b}$$ same value as by the other methods.

Fourth method.

Multiply equation (A) by an indefinite number m,

And $\quad amx+bmy=mc$

Subtract (B) $\quad a'x+b'y=c'$

And $\quad (am-a')x+(bm-b')y=mc-c'.$

Now the value of m may be so assumed as to render the coefficient of $x=0$, or $am-a'=0$.

Then $\quad (bm-b')y=mc-c'$

Or $$y=\frac{mc-c'}{bm-b'} \qquad (C)$$

But $am-a'=0$, or $m=\frac{a'}{a}$.

Put this resultant value of m, in equation (C), and

$$y=\frac{c\times\frac{a'}{a}-c'}{b\times\frac{a'}{a}-b'}=\frac{ca'-ac'}{a'b-ab'}$$

by multiplying both numerator and denominator by a.

(Art. 50.) The principles just explained for elimination between two quantities may be extended to any number, where the number of independent equations given are equal to the number of unknown quantities. For instance, suppose we have the three independent equations:

$$ax+by+cz=d \qquad (A)$$
$$a'x+b'y+c'z=d' \qquad (B)$$
$$a''x+b''y+c''z=d'' \qquad (C)$$

We can eliminate either x, or y, or z, (whichever may be most convenient in any definite problem) between equations (A) and (B,) and we shall have a new equation containing only two unknown quantities. We can then eliminate the same letter between equations (B) and (C,) or (A) and (C,) and have another equation containing the *same two* unknown quantities.

Then we shall have two independent equations, containing two unknown quantities, which can be resolved by either of the four methods already explained.

(Art. 51.) Another theoretical method, from the French, we present to the reader, more for curiosity than for any thing else.

Multiply the first equation (A,) by an indefinite assumed number m. Multiply the second equation (B) by another indefinite number n, and add their products together. Their sum will be

$$(am+a'n)x+(bm+b'n)y+(cm+c'n)z=dm+nd'$$

Subtract eq. (C) $\quad a''x \qquad +b''y \qquad c''z=d''$.

And $\quad (am+a'n-a'')x+(bm+b'n-b'')y+(cm+c'n-c'')z$
$\qquad =dm+nd'-d''$

As m and n are independent and arbitrary numbers, they can be so assumed that

$$am+a'n-a''=0 \quad \text{and} \quad bm+b'n-b''=0.$$

Then $$am+a'n=a'' \quad (1) \quad \text{and} \quad bm+b'n=b'' \quad (2)$$

And $$z=\frac{dm+nd'-d''}{cm+c'n-c''} \quad (3)$$

From equations (1) and (2) we can find the values of m and n, which values may be substituted in equation (3,) and then z will be fully determined.

EXAMPLES FOR PRACTICE.

1. Given $\left\{\begin{array}{c} 8x+5y=68 \\ 12x+7y=100 \end{array}\right\}$ to find the values of x and y.

We can resolve this problem by either one of the four methods just explained. But we would not restrict the pupil to the very letter of the rule, for that in many cases might lead to operations unnecessarily lengthy.

If we take the *third method* of elimination, we should multiply the first equation by 12, the second by 8; but as the coefficients of x contain the common factor 4, we can multiply by 3 and 2, in place of 12 and 8. That is, multiply by the *fourth part* of 12 and 8.

In practice even this form need not be observed, we may decide on our multipliers by inspection only.

Three times the 1st gives $24x+15y=204$

Twice the 2d gives $24x+14y=200$

Difference gives $y=4$

Substituting this value of y in first equation, and

$$8x+20=68 \quad \text{or} \quad x=6.$$

In solving this, we have used *modifications* of the 3d and 2d formal methods.

For exercise, let us use the 4th method.

$$\text{Take}\qquad \begin{array}{r} 8mx+5my=68m \\ 12x+\ 7y\ =100 \\ \hline (8m-12)x+(5m-7)y=68m-100 \end{array}$$

Assume $8m-12=0$.

Then $y=\dfrac{68m-100}{5m-7}$. But $m=\frac{3}{2}$.

Therefore $y=\dfrac{68\times\frac{3}{2}-100}{5\times\frac{3}{2}-7}=\dfrac{204-200}{15-14}=4$ *Ans.*

2. Given $\left\{\begin{array}{l}5x+2y=19\\7x-6y=9\end{array}\right\}$ to find x and y.

If we multiply the first of these equations by 3, the coefficients of y will be equal, and the equations become

$$15x+6y=57,$$

$$\text{And}\qquad 7x-6y=9.$$

To eliminate y, we add these equations (the signs of the terms containing y being unlike), and there results

$$22x=66,$$

$$x=3.$$

This value of x put in the 1st equation gives

$$15+2y=19,$$

$$\text{And}\qquad y=2.$$

3. Given $\dfrac{x+8}{4}+6y=21$ and $\dfrac{y+6}{3}+5x=23$ to find x and y

Clear of fractions and reduce. We then have

$$x+24y=76$$

$$\text{And}\qquad 15x+y=63.$$

In this case there are no abbreviations of the rules, as the coefficients of the unknown terms are *prime* to each other. Continuing the operation, we find $x=4$, $y=3$.

4. Given $x+y=17$ and $\frac{2x}{3}=\frac{3y}{4}$ to find x and y.

Owing to the peculiarity of form in the 2d equation, it is most expedient to resolve this by the 2d method.

From the 2d, $x=\frac{9y}{8}$. Then $\frac{9y}{8}+y=17$.

Clearing of fractions, $9y+8y=17\times 8$.

Or, $17y=17\times 8$, or $y=8$.

Hence, $x=9$.

5. Given $\left\{\begin{matrix}\frac{1}{8}x+8y=194\\ \frac{1}{8}y+8x=131\end{matrix}\right\}$ to find the values of x and y.

Here we observe that both x and y are divided by 8, x in one equation, and y in the other; also, x and y are both multiplied by 8.

(Art. 51.) All such circumstances enable us to resort to many pleasant expedients which go far to teach the true spirit of algebra.

Add these two equations, and $\frac{x+y}{8}+8(x+y)=325$.

Assume $x+y=s$.

Or let s represent the sum of $x+y$, then $\frac{1}{8}s+8s=325$.

Clear of fractions, and $s+64s=325\times 8$.

Unite and divide by 65 and $s=5\times 8$.

Or $x+y=5a$. $(\mathcal{A})$ By returning to the value of s, and putting $a=8$.

Multiply the 1st equation by 8, and

$$\begin{array}{ll} & x+64y=194a \\ \text{Subtract } (\mathcal{A}) & x\ +\ \ y=5a \\ \hline \text{Rem.} & 63y=189a \end{array}$$

Divide by 63 and $y=3a=24$. Whence $x=2a=16$.

Let the pupil take any one of the formal rules for the solution of the preceding equations, and mark the difference.

6. Given $\frac{1}{3}x+3y=21$ and $\frac{1}{3}y+3x=29$ to find x and y.

Ans. $x=9$. $y=6$.

7. Given $4x+y=34$ and $4y+x=16$, to find x and y.
Ans. $x=8$. $y=2$

8. Given $\frac{1}{2}x+\frac{1}{3}y=14$ and $\frac{1}{3}x+\frac{1}{2}y=11$, to find x and y.
Ans. $x=24$. $y=6$

9. Given $x+\frac{1}{2}y=8$ and $\frac{1}{2}x+y=7$ to find x and y.
Ans. $x=6$. $y=4$.

10. Given $\frac{1}{7}x+7y=99$ and $\frac{1}{7}y+7x=51$ to find x and y.
Ans. $x=7$. $y=14$

11. Given $\left\{\begin{array}{l}\frac{1}{2}x-12=\frac{1}{4}y+8\\ \frac{x+y}{5}+\frac{1}{3}x-8=\frac{2y-x}{4}+27.\end{array}\right.$
Ans. $x=60$ $y=40$

12. Given $\frac{a}{x}+\frac{b}{y}=6$ and $\frac{c}{x}+\frac{d}{y}=10$, to find x and y.

Multiply the first equation by c, the second by a, and we shall have

$$\frac{ac}{x}+\frac{bc}{y}=6c$$

$$\frac{ac}{x}+\frac{ad}{y}=10a.$$

By subtraction $(bc-ad)\frac{1}{y}=6c-10a$

Therefore $\frac{bc-ad}{6c-10a}=y.$

13. Given $\frac{147}{x}-\frac{147}{y}=28$ (A) and $\frac{17}{x}+\frac{56}{y}=\frac{41}{3}$ (B) to find the values of x and y.

Divide equation (A) by 7, and $\frac{21}{x}-\frac{21}{y}=4.$

Divide this result by 21, and $\frac{1}{x}-\frac{1}{y}=\frac{4}{21}$ (C)

Multiply (C) by 17, gives $\frac{17}{x}-\frac{17}{y}=\frac{68}{21}$ (D)

Subtract (D) from (B) and we have $\frac{73}{y}=\frac{219}{21}$.

Divide by 73, and $\frac{1}{y}=\frac{3}{21}=\frac{1}{7}$ or $y=7$.

Putting this value in equation (C) and reducing we find $x=3$.

14. Given $\frac{4}{x}+\frac{5}{y}=\frac{9}{y}-1$ and $\frac{5}{x}+\frac{4}{y}=\frac{7}{x}+\frac{3}{2}$ to find the values of x and y

Ans. $x=4$ and $y=2$.

15. Given $\left\{\begin{array}{l} x+150 : y-50 :: 3 : 2 \\ x-50 : y+100 :: 5 : 9 \end{array}\right\}$ to find x and y.

Ans. $x=300$. $y=350$.

16. Given $3x+6y+1=\frac{6x^2-24y^2+130}{2x-4y+3}$

And $3x=\frac{151-16x}{4y-1}+\frac{9xy-110}{3y-4}$ to find x and y.

Ans. $x=9$. $y=2$.

In the first equation actually divide the numerator by the denominator, then drop equals from both sides.

17. Given $\left\{\begin{array}{l} 3x-\frac{1}{2}y=3\frac{1}{2} \\ -x+7y=33 \end{array}\right\}$ to find the values of x and y.

Ans. $x=2$. $y=5$

18. Given $\left\{\begin{array}{l} \frac{1}{2}x+\frac{1}{3}y=8 \\ \frac{1}{3}x-\frac{1}{5}y=-1 \end{array}\right\}$ to find x and y.

Ans. $x=6$. $y=15$.

19. Given $x+y=8$ and $x^2-y^2=16$ to find x and y.

Ans. $x=5$. $y=3$.

20. Given $4(x+y)=9(x-y)$ and $x^2-y^2=36$ to find x and y.

Ans. $x=6\frac{1}{2}$ $y=2\frac{1}{2}$.

21. Given $x:y::4:3$ and $x^3-y^3=37$, to find x and y.
Ans. $x=4$. $y=3$.

22. Given $x+y=a$ and $x^2-y^2=ab$ to find x and y.
Ans. $x=\frac{a+b}{2}$. $y=\frac{a-b}{2}$

23. Given $\frac{x+1}{y}=\frac{1}{3}$ and $\frac{x}{y+1}=\frac{1}{4}$ to find x and y.
Ans. $x=4$. $y=15$

24. Given $\frac{1}{3}(x+2)+8y=31$ and $\frac{1}{4}(y+5)+10x=192$ to find the values of x and y. *Ans.* $x=19$. $y=3$

25. Given $3x+7y=79$ and $2y+\frac{1}{2}x=19$ to find the values of x and y. *Ans.* $x=10$. $y=7$.

26. Given $\frac{1}{2}(x+y)+25=x$ and $\frac{1}{3}(x+y)-5=y$ to find the values of x and y. *Ans.* $x=85$. $y=35$.

27. Given $x-4=y+1$ and $5x-\frac{y}{3}=\frac{3x}{4}-\frac{8y}{6}+37$ to find the values of x and y. *Ans.* $x=8$. $y=3$

28. Given $4-\frac{y-x}{6}=y-17\frac{2}{3}$ and $\frac{y}{5}=\frac{x}{5}+2$ to find the values of x and y. *Ans.* $x=10$. $y=20$.

29. Given $\frac{7x-21}{6}+y-\frac{x}{3}=4+\frac{3x-19}{2}$ and $\frac{2x+y}{2}-\frac{9x-7}{8}=\frac{3y+9}{4}-\frac{4x+5y}{16}$ to find x and y. *Ans.* $x=9$. $y=4$.

30. Given $\left\{\begin{array}{l} x-\frac{2y-x}{23-x}=20-\frac{59-2x}{2} \\ y+\frac{y-3}{x-18}=30-\frac{73-3y}{3} \end{array}\right\}$ to find x and y

Ans. $x=21$. $y=20$.

CHAPTER III.

Solution of Equations involving three or more unknown quantities.

(Art. 52.) No additional principles are requisite to those given in articles 49 and 50.

EXAMPLES.

1. Given $\left\{\begin{array}{l} x+y+z=9 \\ x+2y+3z=16 \\ x+3y+4z=21 \end{array}\right\}$ to find x, y, and z.

By the 1st method, transpose the terms containing y and z in each equation, and

$$x=9-y-z,$$
$$x=16-2y-3z,$$
$$x=21-3y-4z.$$

Then putting the 1st and 2d values equal, and the 2d and 3d values equal, gives

$$9-y-z=16-2y-3z,$$
$$16-2y-3z=21-3y-4z.$$

Transposing and condensing terms, and

$$y=7-2z,$$

Also, $y=5-z,$

Hence, $5-z=7-2z,$ or $z=2$

Having $z=2$, we have $y=5-z=3$, and having the values of both z and y, by the first equation we find $x=4$.

2. Given $\left\{\begin{array}{l} 2x+4y-3z=22 \\ 4x-2y+5z=18 \\ 6x+7y-z=63 \end{array}\right\}$ to find values of x, y and z.

Multiplying the first equation by 2, $4x+8y-6z=44$

And subtracting the second, $4x-2y+5z=18$

The result is, (A) $10y-11z=26$

Then multiply the first equation by 3, $6x+12y-9z=66$
And subtract the third, $6x+7y-z=63$

The result is, (B) $5y-8z=3$

Multiply the new equation (B) by 2, $10y-16z=6$
And subtract this from equation (A) $10y-11z=26$

The result is, $5z=20$
Therefore $z=4$
Substituting the value of z in equation (B) and we find $y=7$.
Substituting these values in the first equation, and we find $x=3$.

3. Given $\left\{\begin{array}{l} 3x+9y+8z=41 \\ 5x+4y-2z=20 \\ 11x+7y-6z=37 \end{array}\right\}$ to find x, y and z.

To illustrate by a practical example we shall resolve this by the principles explained in (Art. 51.)

$$\begin{array}{l} 3mx+9my+8mz=41m \\ 5nx+4ny-2nz=20n \end{array}$$

$$\begin{array}{lllll} \text{Sum} & (3m+5n)x & +(9m+4n)y & +(8m-2n)z & =41m+20n \\ \text{Take} & 11x & +7y & -6z & =37 \end{array}$$

Rem. $(3m+5n-11)x-(7-9m-4n)y+(8m-2n+6)z=41m+20n-37$.

Assume $3m+5n=11$ (1)

And $9m+4n=7$ (2)

Then $z=\dfrac{41m+20n-37}{8m-2n+6}$ (3)

From equations (1) and (2) we find $m=-\frac{3}{11}$ and $n=\frac{26}{11}$
These values substituted in equation (3) we have

$$z=\frac{-41\times\frac{3}{11}+20\times\frac{26}{11}-37}{-8\times\frac{3}{11}-2\times\frac{26}{11}+6}$$

Multiply both numerator and denominator by 11, and we shall have $z=\dfrac{-123+520-407}{-24-52+66}=\dfrac{-10}{-10}=1$.

Putting this value of z in the 1st and 2d equations, we shall have only two equations involving x and y, from which the values of these letters may be determined.

These equations can be resolved with much more facility by multiplying the 2d equation by 4, then adding it to the 1st to destroy the terms containing z.

Afterwards multiplying the 2d equation by 3, and subtracting the 3d equation, and there will arise two equations containing x and y, which mav be resolved by one of the methods already explained.

(Art. 53.) When three, four, or more unknown quantities with as many equations are given, and their coefficients are all *prime* to each other, the operation is necessarily long. But when several of the coefficients are multiples, or *measures* of each other, or unity, several *expedients* may be resorted to for the purpose of facilitating calculation.

No specific rules can be given for mere expedients. Examples alone can illustrate, but even examples will be fruitless to one who neglects general principles and definite theories. Some few expedients will be illustrated by the following

EXAMPLES.

1. Given $\left\{\begin{array}{l} x+y+z=31 \\ x+y-z=25 \\ x-y-z=9 \end{array}\right\}$ to find x, y, and z.

Subtract the 2d from the 1st, and $2z=6$.

Subtract the 3d from the 2d, and $2y=16$.

Add the 1st and 3d, and $2x=40$.

2 Given $\left\{\begin{array}{l} x+y+z=26 \\ x-y \quad\ = 4 \\ x-z \quad\ = 6 \end{array}\right\}$ to find x, y and z.

Add all three, and $3x=36$ or $x=12$.

3. Given $\left\{\begin{array}{l} x-y-z=6 \\ 3y-x-z=12 \\ 7z-y-x=24 \end{array}\right\}$ to find x, y and z

Assume $x+y+z=s$. Add this equation to each of the given equations, and we then have

$$\begin{aligned} 2x&=\ 6+s, && (A)\\ 4y&=12+s, && (B)\\ 8z&=24+s. && (C) \end{aligned}$$

Multiply (A) by 4, and (B) by 2, and we have

$$\begin{aligned} 8x&=24+4s,\\ 8y&=24+2s,\\ 8z&=24+\ s.\\ \hline 8s&=3\times24+7s. \end{aligned}$$

By addition, $8s=3\times24+7s$. Or $s=72$.

Put this value of s in equation (A) and we shall have

$2x=6+72$. Or $x=3+36=39$, &c.

4. Given $x+\frac{1}{2}y=100$, $y+\frac{1}{3}z=100$, $z+\frac{1}{4}x=100$ to find x, y, and z.

Put $a=100$. *Ans.* $x=64$, $y=72$, and $z=84$.

5 Given $\left\{\begin{aligned} u+v+x+y&=10\\ u+v+x+z&=11\\ u+v+y+z&=12\\ u+x+y+z&=13\\ v+x+y+z&=14 \end{aligned}\right\}$ to find the value of each.

Here are *five* letters and five equations. Each letter has the *same* coefficient, *one* understood. Each equation has 4 letters, z is wanting in the 1st equation, y in the 2d, &c.

Now assume $u+v+x+y+z=s$.

Then

$$\begin{aligned} s-z&=10 && (A)\\ s-y&=11\\ s-x&=12\\ s-v&=13\\ s-u&=14\\ \hline 5s-s&=60 \end{aligned}$$

Add, and $5s-s=60$ Or $s=15$.

Put this value of s in equation (A), and $z=5$, &c.

6. Given $x+y=a$, $x+z=b$, $y+z=c$.

Add the 1st and 2d, and from the sum subtract the 3d.

Ans. $x=\frac{1}{2}(a+b-c)$, $y=\frac{1}{2}(a+c-b)$, $z=\frac{1}{2}(b+c-a)$.

7. Given $\left\{\begin{array}{l} 2x=u+y+z \\ 3y=u+x+z \\ 4z=u+x+y \\ u=x-14 \end{array}\right\}$ to find the value of u, x, y, and z.

Ans. $u=26$, $x=40$, $y=30$, $z=24$.

8 Given $\left\{\begin{array}{l} \frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z=62 \\ \frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=47 \\ \frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z=38 \end{array}\right.$ *Ans.* $\left\{\begin{array}{l} x=24. \\ y=60. \\ z=120. \end{array}\right.$

9. Given $\left\{\begin{array}{l} x+a=y+z \\ y+a=2x+2z \\ z+a=3x+3y \end{array}\right.$ *Ans.* $\left\{\begin{array}{l} x=\frac{1}{11}a. \\ y=\frac{5}{11}a \\ z=\frac{7}{11}a. \end{array}\right.$

10. Given $\left\{\begin{array}{r} 2x+y-2z=40 \\ 4y-x+3z=35 \\ 3u+t=13 \\ y+u+t=15 \\ 3x-y+3t-u=49 \end{array}\right.$ *Ans.* $\left\{\begin{array}{l} x=20 \\ y=10. \\ z=5 \\ u=4. \\ t=1. \end{array}\right.$

(Art. 54.) *Problems producing simple equations involving two or more unknown quantities.*

1. Find three numbers such, that the product of the 1st and 2d, shall be 600; the product of the 1st and 3d, 300; and the product of the 2d and 3d, 200.

Ans. The numbers are, 30, 20, and 10.

2. Find three numbers, such that the *first* with $\frac{1}{2}$ the sum of the *second* and *third* shall be 120; the *second* with $\frac{1}{5}$ the difference of the *third* and *first* shall be 70; and the sum of the three numbers shall be 190. *Ans.* 50; 65; 75.

3. A certain sum of money was to be divided among three persons, *A*, *B*, and *C*, so that *A*'s share exceeded $\frac{4}{7}$ of the shares of *B* and *C* by \$120; also the share of *B* exceeded $\frac{3}{8}$ of the shares of *A* and *C* by \$120; and the share of *C*, likewise, exceeded $\frac{2}{9}$ of the shares of *A* and *B* by \$120. What was each person's share?

Ans. *A*'s share, \$600; *B*'s, 480; and *C*'s 360.

4. *A* and *B*, working at a job, can earn \$40 in 6 days; *A* and *C* together can earn \$54 in 9 days; and *B* and *C* \$80 in 15 days. What can each person alone earn in a day?

Let *A* earn x, *B* earn y, and *C* earn z dollars per day, then,

By the question, $6x+6y=40$

$$9x+9z=54$$

$$15y+15z=80$$

Dividing the equations by the coefficients of the unknown quantities, we have,

$$x+y=6\tfrac{2}{3}$$

$$x+z=6$$

$$y+z=5\tfrac{1}{3}.$$

See Problem 6. (Art. 53.)

A man has 4 sons. The sum of the ages of the first, second and third is 18 years; the sum of the ages of the first, second and fourth is 16 years; the sum of the ages of the first, third and fourth is 14 years; the sum of the ages of the second, third and fourth is 12 years. What are their ages? See Problem 5. (Art. 53.) *Ans.* Their ages are, 8, 6, 4, 2.

5. *A*, *B* and *C* sat down to play, each one with a certain number of shillings; *A* loses to *B* and *C* as many shillings as each of them has. Next *B* loses to *A* and *C* as many as each of them now has. Lastly, *C* loses to *A* and *B* as many as each of them now has. After all, each one of them has 16 shillings. How much did each one gain or lose?

Let $x=$ the number of shillings *A* had at first
$y=$ *B*'s shillings, and
$z=$ *C*'s shillings.

Then, by resolving the problem, we shall find $x=26$, $y=14$ and $z=8$. Therefore, *A* lost 10 shillings, *B* gained 2, and *C* 8.

N. B. When the equations are found, divide the 1st by 4, the 2d by 2, and then compare them with Ex. 3. (Art. 53.)

6. A gentleman left a sum of money to be divided among four servants, so that the share of the first was $\frac{1}{2}$ the sum of the shares of the other three; the share of the second, $\frac{1}{3}$ of the sum of the other three; and the share of the third, $\frac{1}{4}$ the sum of the other three; and it was found that the share of the last was 14 dollars less than that of the first. What was the amount of money divided, and the shares of each respectively?

Ans. The sum was \$120; the shares 40, 30, 24 and 26.

Observe Prob. 7. (Art. 53,) in connection with this problem.

7. A jockey has two horses, and two saddles which are worth 15 and 10 dollars, respectively. Now if the better saddle be put on the better horse, the value of the better horse and saddle would be worth $\frac{4}{3}$ of the other horse and saddle. But if the better saddle be put on the poorer horse, and the poorer saddle on the better horse, the value of the better horse and saddle is worth once and $\frac{2}{13}$ the value of the other. Required the worth of each horse? *Ans.* 65 and 50 dollars.

8. A merchant finds that if he mixes sherry and brandy in quantities which are in proportion of 2 to 1, he can sell the mixture at 78 shillings *per dozen;* but if the proportion be 7 to 2 he can sell it at 79 shillings *per dozen.* Required the price per dozen of the sherry and of the brandy?

Ans. Sherry, 81*s.* Brandy, 72*s.*

In the solution of this question, put $a=78$. Then $a+1=79$.

9. Two persons, *A* and *B*, can perform a piece of work in 16 days. They work together for four days, when *A* being called off, *B* is left to finish it, which he does in 36 days. In what time would each do it separately?

Ans. *A* in 24 days; *B* in 48 days.

10 What fraction is that, whose numerator being doubled, and denominator increased by 7, the value becomes $\frac{2}{3}$; but the denominator being doubled, and the numerator increased by 2, the value becomes $\frac{3}{5}$? *Ans.* $\frac{4}{5}$.

11. Two men wishing to purchase a house together, valued at 240 (a) dollars; says *A* to *B*, if you will lend me $\frac{2}{3}$ of your money I can purchase the house alone; but says *B* to *A*, if you lend me $\frac{3}{4}$ of yours, I can purchase the house. How much money had each of them? *Ans.* *A* had \$160. *B* \$120.

12. It is required to divide the number 24 into two such parts, that the quotient of the greater part divided by the less, may be to the quotient of the less part divided by the greater, as 4 to 1

Ans. 16 and 8.

13. A certain company at a tavern, when they came to settle their reckoning, found that had there been 4 more in company, they might have paid a shilling a-piece less than they did; but that if there had been 3 fewer in company, they must have paid a shilling a-piece more than they did. What then was the number of persons in company, and what did each pay?

Ans. 24 persons, each paid 7*s*.

14. There is a certain number consisting of two places, a unit and a ten, which is four times the sum of its digits, and if 27 be added to it, the digits will be inverted. What is the number?

Ans. 36.

Note. Undoubtedly the reader has learned in arithmetic that numerals have a specific and a local value, and every remove from the unit multiplies by 10. Hence, if x represents a digit in the place of tens, and y in the place of units, the number must be expressed by $10x+y$. A number consisting of three places, with x, y and z to represent the digits, must be expressed by $100x+10y+z$.

15. A number is expressed by three figures; the sum of these figures is 11; the figure in the place of units is double that in the place of hundreds, and when 297 is added to this number, the sum obtained is expressed by the figures of this number reversed. What is the number? *Ans.* 326.

16. Divide the number 90 into three parts, so that twice the first part increased by 40, three times the second part increased by 20, and four times the third part increased by 10, may be all equal to one another.

Ans. First part 35, second 30, and third 25.

17. A person who possessed \$100,000 (a,) placed the greater part of it out at 5 per cent interest, and the other part at 4 per cent. The interest which he received for the whole amounted to 4640 (b) dollars. Required the two parts.

Ans. 64,000 and 36,000 dollars.

General Answer. ($100b-4a$) for the greater part, and ($5a-100b$) for the less.

18. A person put out a certain sum of money at interest at a certain rate. Another person put out \$10,000 *more*, at a rate 1 per cent. higher, and received an income of \$800 more. A third person put out \$15,000 more than the first, at a rate 2 per cent. higher, and received an income greater by \$1,500. Required the several sums, and their respective rates of interest.

Ans. Rates 4, 5 and 6 per cent. Capitals \$30,000, \$40,000 and \$45,000.

19. A widow possessed 13,000 dollars, which she divided into two parts and placed them at interest, in such a manner, that the incomes from them were equal. If she had put out the first portion at the same rate as the second, she would have drawn for this part 360 dollars interest, and if she had placed the second out at the same rate as the first, she would have drawn for it 490 dollars interest. What were the two rates of interest?

Ans. 7 and 6 per cent.*

20. There are three persons, *A*, *B* and *C*, whose ages are as follows: if *B*'s age be subtracted from *A*'s, the difference will be *C*'s age; if five times *B*'s age and twice *C*'s age, be added together, and from their sum *A*'s age be subtracted, the remainder will be 147. The sum of all their ages is 96. What are their ages?

Ans. *A*'s 48, *B*'s 33, *C*'s 15.

21. Find what each of three persons, *A*, *B* and *C*, is worth, from knowing, 1st, that what *A* is worth added to 3 times what *B* and *C* are worth, make 4700 dollars; 2d, that what *B* is worth added to four times what *A* and *C* are worth make 5800 dollars; 3d, that what *C* is worth added to five times what *A* and *B* are worth make 6300 dollars.

Ans. *A* is worth 500, *B* 600, *C* 800 dollars.

See brief solution to these two problems, 18 and 19, in Key to Algebra.

Put $s=$ the sum that *A*, *B* and *C* are worth, to make an auxiliary equation.

22. Find what each of three persons, *A*, *B*, *C*, is worth, knowing, 1st, that what *A* is worth added to l times what *B* and *C* are worth is equal to p; 2d, that what *B* is worth added to m times what *A* and *C* are worth is equal to q; 3d, that what *C* is worth added to n times what *A* and *B* are worth is equal to r.

Let $x=A$'s capital, $y=B$'s, and $z=C$'s.
Then $x+ly+lz=p$, is the first equation.

Assume $x+y+z=s$. Multiply this equation by l and subtract the former, and $(l-1)x=ls-p$.

$$\left.\begin{array}{lr} \text{Or} & x=\dfrac{ls-p}{l-1} \\ \text{By a similar operation,} & y=\dfrac{ms-q}{m-1} \\ \text{And,} & z=\dfrac{ns-r}{n-1} \end{array}\right\} \quad \text{(A)}$$

By addition, $$s=\frac{ls-p}{l-1}+\frac{ms-q}{m-1}+\frac{ns-r}{n-1}.$$

This equation may take the following form:

$$s=\left(\frac{l}{l-1}+\frac{m}{m-1}+\frac{n}{n-1}\right)s-\left(\frac{p}{l-1}+\frac{q}{m-1}+\frac{s}{n-1}\right)$$

Now as the terms in parenthesis are fully determined, of known value, we may represent the first by a, the second by b, and this last form becomes

$$s=as-b$$

By transposition, &c. $(a-1)s=b$

Therefore $$s=\frac{b}{a-1}$$

This known value of s put in each of the equations marked (A), and the values of x, y and z will be theoretically determined.

23. Three brothers made a purchase of $2000 ($a$;) the first wanted in addition to his own money $\frac{1}{2}$ the money of the second, the second wanted in addition to his own $\frac{1}{3}$ of the money of the third, and the third required in addition to his own $\frac{1}{4}$ of the money of the first. How much money had each?

Ans. 1st had, $1280; 2d, $1440; and the 3d, $1680.

Gen. Ans. 1st had $\frac{16}{25}a$; 2d, $\frac{18}{25}a$; and the 3d. $\frac{21}{25}a$.

See Prob. 6. (Art. 53.)

24. Some hours after a courier had been sent from *A* to *B*, which are 147 miles distant, a second was sent, who wished to overtake him just as he entered *B*, and to accomplish this he must perform the journey in 28 hours less time than the first did Now the time that the first travels 17 miles added to the time the second travels 56 miles is $13\frac{2}{3}$ hours. How many miles does each go per hour? *Ans.* 1st 3, the 2d, 7 miles per hour.

25. There are two numbers, such that $\frac{1}{2}$ the greater added to $\frac{1}{3}$ the lesser, is 13; and if $\frac{1}{2}$ the lesser is taken from $\frac{1}{3}$ the greater, the remainder is nothing. Required the numbers.

Ans. 18 and 12.

26. Find three numbers of such magnitude, that the 1st with the $\frac{1}{2}$ sum of the other two, the second with $\frac{1}{3}$ of the other two, and the third with $\frac{1}{4}$ of the other two, may be the same, and amount to 51 in each case. *Ans.* 15, 33, and 39.

27. *A* said to *B* and *C*, "Give me, each of you, 4 of your sheep, and I shall have 4 more than you will have left." *B* said to *A* and *C*, "If each of you will give me 4 of your sheep, I shall have twice as many as you will have left." *C* then said to *A* and *B*, "Each of you give me 4 of your sheep, and I shall have three times as many as you will have left." How many had each? *Ans.* *A* 6, *B* 8, and *C* 10.

28. What fraction is that, to the numerator of which if 1 be added, the fraction will be $\frac{1}{3}$: but if 1 be added to the denominator, the fraction will be $\frac{1}{4}$? *Ans.* $\frac{4}{15}$.

29. What fraction is that, to the numerator of which if 2 be added, the fraction will be $\frac{5}{7}$; but if 2 be added to the denominator, the fraction will be $\frac{1}{3}$? *Ans.* $\frac{3}{7}$.

30. What fraction is that whose numerator being doubled, and its denominator increased by 7, the value becomes $\frac{2}{3}$; but the denominator being doubled, and the numerator increased by 2, the value becomes $\frac{3}{5}$? *Ans.* $\frac{4}{5}$.

31. If *A* give *B* \$5 of his money, *B* will have twice as much money as *A* has left; and if *B* give *A* \$5, *A* will have thrice as much as *B* has left. How much had each?

Ans. *A* \$13, and *B* \$11.

32. A corn factor mixes wheat flour, which cost him 10 shillings per bushel, with barley flour, which cost 4 shillings per bushel, in such proportion as to gain 43¾ per cent. by selling the mixture at 11 shillings per bushel. Required the proportion.

Ans. The proportion is 14 bushels of wheat flour to 9 of barley.

33. There is a number consisting of two digits, which number divided by 5 gives a certain quotient and a remainder of *one*, and the same number divided by 8 gives another quotient and a remainder of *one*. Now the quotient obtained by dividing by 5 is double of the value of the digit in the ten's place, and the quotient obtained by dividing by 8 is equal to 5 times the unit digit. What is the number? *Ans.* 41.

Interpretation of negative values resulting from the solution of equations.

(Art. 55.) The resolution of proper equations drawn from problems not only reveal, the numeral result, but improper enunciation by the change of signs. Or the signs being true algebraic language, they will point out errors in relation to terms in common language, as the following examples will illustrate:

1. The sum of two numbers is 120, and their difference is 160; what are the numbers?

Let x be the greater and y the less. Then

$$x+y=120 \qquad (1)$$
$$x-y=160 \qquad (2)$$

The solution gives $x=140$, and $y=-20$.

Here it appears that one of the numbers is greater than the sum given in the enunciation, yet the sum of x and y, in the algebraic sense, is 120.

There is no such *abstract* number as -20, and when minus appears it is only *relative* or opposite in direction or condition to plus, and the problem is susceptible of interpretation in an algebraic sense, but not in a definite arithmetical sense.

Indeed we might have determined this at once by a consideration of the problem, for the difference of the two numbers is given, greater than their sum. But we can form a problem, an algebraic (not an abstract) problem that will exactly correspond with these conditions, thus:

The joint property of two men amounts to 120 dollars, and one of them is worth 160 dollars more than the other. What amount of property does each possess?

The answer must be $+140$ and -20 dollars; but there is no such thing as minus $20 in the abstract; it must be interpreted *debt*, an opposite term to positive money in hand.

2. Two men, A and B, commenced trade at the same time; A had **3** times as much money as B, and continuing in trade, A gains 400 dollars, and B 150 dollars; now A has twice as much money as B. How much did each have at first?

Without any special consideration of the question, it implies that both had money, and asks how much. But on resolving the question with x to represent A's money, and y B's, we find

$$x=-300$$

$$\text{And} \quad y=-100 \text{ dollars.}$$

That is, they had no money, and the minus sign in this case indicates *debt;* and the solution not only reveals the numerical values, but the true conditions of the problem, and points out the necessary corrections of language to correspond to an arithmetical sense, thus:

A *is three times as much in debt as* B; *but* A *gains* 400 *dollars, and* B 150; *now* A *has twice as much money as* B. *How much were each in debt?*

As the *enunciation* of this problem corresponds with the real circumstance of the case, we can resolve the problem without a minus sign in the result. Thus:

Let $x=$ *B*'s debt, then $3x=$ *A*'s debt
$150-x=$ *B*'s money, $400-3x=$ *A*'s money
Per question, $400-3x=300-2x$. Or $x=100$.

3. What number is that whose fourth part exceeds its third part by 12? *Ans.* —144.

But there is no such abstract number as —144, and we cannot interpret this as *debt*. It points out error or *impossibility*, and by returning to the question we perceive that a fourth part of any number whatever cannot exceed its third part; it must be, its third part exceeds its fourth part by 12, and this enunciation gives the positive number, 144. Thus do equations rectify *subordinate* errors, and point out special conditions.

4. A man when he was married was 30 years old, and his wife 15. How many years must elapse before his age will be three times the age of his wife?

Ans. The question is incorrectly enunciated; $7\frac{1}{2}$ years *before* the marriage, *not* after, their ages bore the specified relation.

5. A man worked 7 days, and had his son with him 3 days, and received for wages 22 shillings. He afterwards worked 5 days, and had his son with him one day, and received for wages 18 shillings. What were his daily wages, and the daily wages of his son?

Ans. The father received 4 shillings per day, and paid 2 shillings for his son's board.

6. A man worked for a person ten days, having his wife with him 8 days, and his son 6 days, and he received $10.30 as compensation for all three; at another time he wrought 12 days, his wife 10 days, and son 4 days, and he received $13.20; at another time he wrought 15 days, his wife 10 days, and his son 12 days, at the same rates as before, and he received $13.85. What were the daily wages of each?

Ans. The husband 75 cts., wife 50 cts. The son 20 cts. *expense* per day.

7. A man wrought 10 days for his neighbor, his wife 4 days, and son 3 days, and received \$11.50; at another time he served 9 days, his wife 8 days, and his son 6 days, at the same rates as before, and received \$12.00; a third time he served 7 days, his wife 6 days, and his son 4 days, at the same rates as before, and he received \$9.00. What were the daily wages of each?

Ans. Husband's wages, \$1.00; wife 0; son 50 cts.

8. What fraction is that which becomes $\frac{3}{5}$ when one is added to its numerator, and becomes $\frac{5}{7}$ when 1 is added to its denominator?

Ans. In an arithmetical sense, there is no such fraction. The algebraic expression, $\frac{-10}{-15}$, will give the required results.

(Art. 58.) By the aid of algebraical equations, we are enabled not only to resolve problems and point out defects or errors in their enunciation, as in the last article, but we are also enabled to demonstrate theorems, and elucidate many philosophical truths The following are examples:

Theorem 1. It is required to demonstrate, that the half sum plus half the difference of two quantities give the greater of the two, and the half sum minus the half difference give the less

Let $x=$ the greater number, $y=$ the less,

$s=$ their sum, $d=$ their difference.

Then $x+y=s$ (A)

And $x-y=d$ (B)

By addition, $2x=s+d$

Or $x=\frac{1}{2}s+\frac{1}{2}d$

Subtract (B) from (A) and divide by 2, and we have

$$y=\tfrac{1}{2}s-\tfrac{1}{2}d$$

These last two equations, which are manifestly true, demonstrate the theorem.

Theorem 2. Four times the product of any two numbers, is equal to the square of their sum, diminished by the square of their difference.

Let $x=$ the greater number, and $y=$ the less, as in the last theorem.

$$\begin{array}{rl} & 2x=s+d \\ \text{And} & \underline{2y=s-d} \end{array}$$

By multiplication $4xy=s^2-d^2$ a demonstration of the theorem.

Many other theorems are demonstrable by algebra, but we defer them for the present, as some of them involve quadratic equations, which have not yet been investigated; and we close the subject of simple equations by the following quite general problem in relation to *space*, *time* and *motion*.

To present it at first, in the most simple and practical manner, let us suppose

Two couriers, A *and* B, 100 *miles asunder on the same road set out to meet each other,* A *going* 6 *miles per hour and* B 4. *How many hours must elapse before they meet, and how far will each travel?*

Let $x=$ *A*'s distance, $y=$ *B*'s, and $t=$ the time.

Then $x+y=100$ (1)

As the miles per hour multiplied by the hours must give the distance each traveled, therefore,

$$x=6t \quad \text{and} \quad y=4t \qquad (2)$$

Substitute these values in equation (1) and

$$(6+4)t=100$$

Therefore, $$t=\frac{100}{6+4} \qquad (3)$$

And $$x=6t=\frac{100\times 6}{6+4} \qquad y=4t=\frac{100\times 4}{6+4} \qquad (4)$$

From equation (3,) we learn that the time elapsed before the couriers met was the whole distance divided by their joint motion per hour, a result in perfect accordance with reason. From equations (4,) we perceive that the distance each must travel is the whole distance asunder multiplied by their respective motions and divided by the sum of their hourly motions.

Now let us suppose the couriers start as before, but travel in the same direction, the one in pursuit of the other. B *having*

100 *miles the start, traveling four miles per hour, pursued by* A, *traveling* 6 *miles per hour. How many hours must elapse before they come together, and what distance must each travel?*

Take the same notation as before.

Then $x-y=100$ (1.) As *A* must travel 100 miles more than *B.* But equations (2,) that is, $x=6t$ and $y=4t$, are true under all circumstances.

$$\text{Then} \qquad (6-4)t=100$$

$$\text{And} \qquad t=\frac{100}{6-4}=50$$

The result in this case is as obvious as an axiom. *A* has 100 miles to gain, and he gains 2 miles per hour, it will therefore require 50 hours.

But it is the *precise form* that we wish to observe. It is the fact that the given distance divided by the difference of their motions gives the time, and their respective distances must be this *time* multiplied by their respective rates of motion.

Now the smaller the difference between their motions, the longer the time before one overtakes the other; when the difference is very small, the time will be very great; when the difference is nothing, the time will be infinitely great; and this is in perfect accordance with reason; for when they travel equally fast one cannot gain on the other, and they can never come together.

If the foremost courier travels faster than the other, they must all the while become more and more asunder; and if they have ever been together it was preceding their departure from the points designated, and in an opposite direction from the one they are traveling, and would be pointed out by a negative result.

(Art. 59.) Let us now make the problem general.

Two couriers, A *and* B, d *miles asunder on the same road, set out to meet each other;* A *going* a *miles per hour,* B *going* b *miles per hour. How many hours must elapse before they meet, and how far will each travel?*

Taking the same notation as in the particular case,

Let $x=$ *A*'s distance, $y=$ *B*'s, and $t=$ the time.

Then $x+y=d$ (1) $x=at$ $y=bt$ (2)

Therefore $(a+b)t=d$ Or $t=\dfrac{d}{a+b}$ (3)

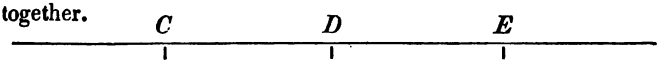
And $x=at=\dfrac{ad}{a+b}$ $y=bt=\dfrac{bd}{a+b}$ (4)

If $a=b$, then $x=\frac{1}{2}d$ and $y=\frac{1}{2}d$. A result perfectly obvious, the rates being equal. Each courier must pass over one half the distance before meeting.

If $a=0$ $x=\dfrac{0\times d}{0+b}=0$ and $y=\dfrac{b\,d}{b}=d$. That is, one will be at rest, and the other will pass over the whole distance.

(Art. 60.) *Now let us consider the other case, in which one courier pursues the other, starting at the same time from different points.*

Let the line *CD* represent the space the couriers are asunder when the pursuit commences, and the point *E* where they come together.



The direction from *C* towards *D* we call *plus*, the other direction will therefore be *minus*.

Now as in the 2d example, (Art. 58.)

Put $x=CE=$ *A*'s distance

$y=DE=$ *B*'s distance

Then $x-y=CD=d$ (1)

As before, let $t=$ the time. Then

$x=at$ $y=bt$ (2)

Therefore $at-bt=d$

And $t=\dfrac{d}{a-b}$ (3)

For the distances we have

$x=\dfrac{ad}{a-b}$ (4) and $y=\dfrac{bd}{a-b}$ (5)

By an examination of these equations, it will be perceived that x and y will be *equal* when a is equal to b, yet d still exists as a difference between them. This is in consequence of x and y in that case being so very great that d is lost in comparison. So all values are great or small only in comparison with others or with our scale of measure.

To make this clear, let us suppose two numbers differ by *one* and if the numbers are small, the difference may be regarded as considerable; if large, more inconsiderable; if still larger, still more inconsiderable, &c. If the numbers or quantities be infinitely great, the comparative small quantity may be rejected. Thus:

5 and 6 differ by 1, and their relation is as 1 to 1.2.

Also, 50 and 51 differ by 1, and their relation is as 1 to 1.02. 500 to 501 are as 1 to 1.002, &c. The relation becoming nearer and nearer equality as the numbers become larger, and when the numbers become infinitely great the difference is comparatively nothing.

When $a=b$ $a-b=0$ and $x=\frac{ad}{0}$ a symbol of infinity. If we suppose b greater than a, $a-b$ will become negative, and as x and y refer to the same point, that point must then be in the backward direction from that we suppose the couriers are moving, and will show how far they have traveled since that event.

If in the equations (3), (4) and (5), $d=0$, and at the same time $a=b$, then we shall have $t=\frac{0}{0}$, $x=\frac{0}{0}$ and $y=\frac{0}{0}$: which shows that $\frac{0}{0}$ is a symbol of indetermination, it being equal to several quantities at the same time. If $d=0$ the two couriers were together at commencement; and if they travel in the same direction, and equally fast, they will be together all the while, and the distances represented by x and y will be equal, and of all possible values. *Hence* $\frac{0}{0}$ *may be taken of any value what-*

ever, and may be made to take a particular value, to correspond to any other circumstance or condition.[*]

APPLICATION.

(Art. 61.) We have hitherto considered *CD* a right line; but the equations would be equally true, if we consider *CD* to be *curved*, and indeed we can conceive the line *CD* to wind about a perfect circle just forming its circumference, and the point *E* upon the circle, *CE* being a little more than one circumference.

This being understood, Equation (3,) (Art. 60,) gives us a solution to the following problems.

1. *The hour and minute hands of a clock are together at* 12 *o'clock. When are they next together?*

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[*] The 26th equation (Art. 40), if resolved in the briefest manner, will show the influence of the factor $\frac{0}{0}$. In the equation referred to, add 30 to both members and divide the numerator of the second member by its denominator, and we have $\frac{5x+5}{x+2}+1=6$. Drop 1, and divide both members by 5, we then have $\frac{x+1}{x+2}=1$, or $x+1=x+2$. Hence $1=2$, a manifest absurdity But all our operations, yea, and all our reasonings have been correct, but we did not pay sufficient attention to dividing the numerator by the denominator, which was $\frac{6(x-2)}{(x-2)}$. Taking 6 for the quotient, which it would be in every case except when $x-2=0$, leads to the absurdity; which absurdity, *in turn*, shows that $x-2=0$, or $x=2$.

As another illustration of the influence of this symbol, take the identical equation $100=100$, or any other similar one.

This is the same as $96+4=96+4$

Transposing, $4-4=96-96$

Resolving into factors, $1(4-4)=24(4-4)$

Dividing by the common factor, and $1=24$;

But, to restore equality, $\frac{0}{0}$ in this case must equal $\frac{1}{24}$, or 24.

Hence we perceive that $\frac{0}{0}$ is indeterminate, in the abstract, but may be rendered definite in particular cases.

By the equation, $t=\frac{d}{a-b}$ this problem and all others like it are already resolved. All we have to do is to determine the values of d, a, and b.

There are 12 spaces (hours) round the dial plate of a clock; hence d may represent 12. a and b are the relative motions of the hands. a moves 12 spaces or entirely round the dial plate while b moves one space. Hence $a=12$, $b=1$, and $a-b=11$.

Consequently, $t=\frac{12}{11}=1\text{h. }5\text{m. }27\frac{3}{11}\text{s.}$

Again. We may demand what time the hour and minute hands of a clock are together between 3 *and* 4.

From 12 o'clock to past 3 o'clock there are 3 revolutions to pass over in place of one, and the solution is therefore $t=\frac{3\times 12}{11}$ and so on for any other hour.

2. *What time between two and three o'clock will the hour and minute hands of a clock make right angles with each other?*

Here the space that the one courier must gain on the other is *two revolutions and a quarter*, or $2\frac{1}{4}d$.

Hence $t=\frac{12}{11}\times 2\frac{1}{4}=\frac{12}{11}\times\frac{9}{4}=\frac{27}{11}=2\text{h. }27\frac{3}{11}\text{m.}$

3. *What time between* 5 *and* 6 *will the two hands of a clock make a right line?*

Here one courier must gain $5\frac{1}{2}$ revolutions, or d in the equation must be multiplied by $5\frac{1}{2}=\frac{11}{2}$.

Hence, $t=\frac{12}{11}\times\frac{11}{2}=6\text{h.}$

That is, the hands make a right line at 6 o'clock, a result manifestly true.

This simple equation enables us to determine the exact time when the two hands of a clock shall be in *any given position.*

We may apply this equation to a large circle, as well as to a small one; it may apply to the apparent circular course of the heavens, as well as to a dial plate of a clock; and the application is equally simple.

The circle of the heavens, like all other circles, is divided into

360 degrees; and the sun and moon apparently follow each other like two couriers round the circle.

In one day the moon moves on an average 13°.1764, (divisions of the circle,) and the sun apparently 0°.98565, or not quite one division of the circle. The moon's motion being most rapid, corresponds to a in the equation, and the sun's apparent motion to b. Then $a-b=13°{:}1764-0.98565=12°.19075$; and the time required for one courier to gain on the other the required space, in this case a revolution of 360 degrees, or $t=\frac{d}{a-b}=\frac{360}{12.19075}$ which gives 29.5305887 days, or 29 days, 12 hours, 44 minutes, 3 seconds, which is the mean time from one change of the moon to another, called a synodic revolution.

These relative apparent motions of the sun and moon round the circular arc of the heavens, are very frequently compared to the motions of the hour and minute hands of a clock round the dial plate; and from the preceding application of the same equation we see how truly.

We may not only apply this equation to the mean motions of the sun and moon, but it is equally applicable to the mean motions of any two planets as seen from the sun. To appearance, the two planets would be nothing more than two couriers moving in a circle, the one in pursuit of the other, and the time between two intervals of coming together, (or coming in conjunction, as it is commonly expressed,) will be invariably represented by the equation

$$t=\frac{d}{a-b}$$

To apply this to the motion of two planets, we propose the following problem:

The planet Venus, as seen from the sun, describes an arc of 1° 36′ *per day, and the earth, as seen from the same point, describes an arc of* 59′. *At what interval of time will these two bodies come in a line with the sun on the same side?*

Here $a=1°36'$ $b=59'$ $d=360°$

Therefore, $a-b=37'$; and as the denominator is minutes, the numerator must be reduced to minutes also; hence the equation becomes

$$t=\frac{360\times 60}{37}=583.8 \text{ days, nearly.}$$

We have not been very minute, as the motions of the planets are not perfectly uniform, and the actual interval between successive conjunctions is slightly variable. Hence we were not particular to take the values of a and b to the utmost fraction. A more rigid result would have been 583.92 days. Half of this time is the interval that *Venus* remains a morning and an evening star.

(Art. 63.) This equation, as simple as it may appear, is one practical illustration of the true spirit and utility of analysis by algebra.

The principles and relations of time and motion are fixed and invariable, and the equation $t=\frac{d}{a-b}$ stands conspicuously as one of these relations.

If t can be determined by observation, as it may be in respect to the earth and the superior planets, the mean daily motions of the planets can be determined; as $d=360°$, $a=59'\ 08''$ the mean motion of the earth, and suppose b the motion of *Mars*, for example, to be unknown.

When unknown, represent it by x.

Then $t=\frac{d}{a-x}$ or $at-tx=d$.

Therefore $x=\frac{at-d}{t}$.

SECTION III.

INVOLUTION

CHAPTER I.

(Art. 64.) Equations, and the resolution of problems producing equations, do not always result in the first powers of the unknown terms, but different powers are frequently involved, and therefore it is necessary to investigate methods of resolving equations containing higher powers than the first; and preparatory to this we must learn involution and evolution of algebraic quantities.

(Art. 65.) Involution is the method of raising any quantity to a given power. Evolution is the reverse of involution, and is the method of determining what quantity raised to a proposed power will produce a given quantity.

As in arithmetic, involution is performed by multiplication, and evolution by the extraction of roots.

The first power is the root or quantity itself;

The second power, commonly called the *square*, is the quantity multiplied by itself;

The third power is the product of the second power by the quantity;

The fourth power is the third power multiplied into the quantity, &c.

The second power of a is $a \times a$ or a^2
The third power is $a^2 \times a$ or a^3
The fourth power is $a^3 \times a$ or a^4
The second power of a^4 is $a^4 \times a^4$ or a^8
The third power of a^4 is $a^8 \times a^4$ or a^{12}

The nth power of a^4 has the exponent 4 repeated n times, or a^{4n}. Therefore, to raise a simple literal quantity to any power, *multiply its exponent by the index of the required power.*

Raise x to the 5th power. The exponent is 1 understood, and this 1 multiplied by 5 gives x^5 for the 5th power.

Raise x^3 to the 4th power	*Ans.* x^{12}.
Raise y^7 to the third power.	*Ans.* y^{21}.
Raise x^n to the 6th power.	*Ans.* x^{6n}.
Raise x^n to the *m*th power.	*Ans.* x^{mn}.
Raise ax^2 to the 3d power.	*Ans.* a^3x^6.
Raise ab^2x^4 to the 2d power.	*Ans.* $a^2b^4x^8$.
Raise c^2y^4 to the 5th power.	*Ans.* $c^{10}y^{20}$.

(Art. 66.) By the definition of powers the second power is any quantity multiplied by itself; hence the second power of ax is a^2x^2, the second power of the coefficient a, as well as the other quantity x; but a may be a numeral, as $6x$, and its second power is $36x^2$. *Hence, to raise any simple quantity to any power, raise the numeral coefficient, as in arithmetic, to the required power, and annex the powers of the given literal quantities*

EXAMPLES.

1. Required the 3d power of $3ax^2$.	*Ans.* $27a^3x^6$.
2. Required the 4th power of $\frac{2}{3}y^2$.	*Ans.* $\frac{16}{81}y^8$.
3. Required the 3d power of $-2x$.	*Ans.* $-8x^3$.
4. Required the 4th power of $-3x$.	*Ans.* $81x^4$.

Observe, that by the rules laid down for multiplication, the *even* powers of minus quantities must be *plus*, and the odd powers *minus*.

5. Required the 2d power of $\frac{2a^2b^6}{5c}$.	*Ans.* $\frac{4a^4b^{12}}{25c^2}$.
6. Required the 6th power of $-\frac{2a}{3x}$.	*Ans.* $\frac{64a^6}{729x^6}$.
7. Required the 6th power of $\frac{a^2b}{\frac{1}{3}x}$.	*Ans* $\frac{729a^{12}b^6}{x^6}$.

(Art. 67.) The powers of compound quantities are raised by the mere application of the rule for compound multiplication. (Art. 12.)

Let $a+b$ be raised to the 2d, 3d, 4th, &c. powers.

$$\begin{array}{ll}
 & a+b \\
 & a+b \\ \cline{2-2}
 & a^2+ab \\
 & \qquad ab+b^2 \\ \cline{2-2}
\text{2d power or square,} & a^2+2ab+b^2 \\
 & a+b \\ \cline{2-2}
 & a^3+2a^2b+ab^2 \\
 & \qquad a^2b+2ab^2+b^3 \\ \cline{2-2}
\text{3d power or cube,} & a^3+3a^2b+3ab^2+b^3 \\
 & a+b \\ \cline{2-2}
 & a^4+3a^3b+3a^2b^2+ab^3 \\
 & \qquad a^3b+3a^2b^2+3ab^3+b^4 \\ \cline{2-2}
\text{The 4th power,} & a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\
 & a+b \\ \cline{2-2}
 & a^5+4a^4b+6a^3b^2+4a^2b^3+ab^4 \\
 & \qquad a^4b+4a^3b^2+6a^2b^3+4ab^4+b^5 \\ \cline{2-2}
\text{The 5th power,} & a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5 \\ \cline{2-2}
 & \qquad\qquad\qquad\qquad \text{\&c., \&c.}
\end{array}$$

By inspecting the result of each product, we may arrive at general principles, according to which any power of a binomial may be expressed, without the labor of actual multiplication. This theorem for abbreviating powers, and its general application to both powers and roots, first shown by Sir Isaac Newton, has given it the name of Newton's binomial, or the *binomial theorem*.

OBSERVATIONS.

Observe the 5th power: a, being the first, is called the leading term; and b, the second, is called the following term. The *sum* of the exponents of the two letters in each and all of the terms amount to the index of the power. In the 5th power, the sum of the exponents of a and b is 5; in the 4th power it is 4; in the 10th power it would be 10, &c. In the 2d power there are *three* terms; in the 3d power there are 4 terms; in the 4th power there are 5 terms; always *one more* term than the index of the power denotes.

The 2d letter does not appear in the first term; the 1st letter does not appear in the last term.

The highest power of the leading term is the index of the given power, and the powers of that letter decrease by *one* from term to term. The second letter appears in the 2d term, and its exponent increases by one from term to term as the exponent of the other letter decreases.

The 8th power of $(a+b)$ is indicated thus: $(a+b)^8$. When expanded, its literal part, (according to the preceding observations) must commence with a^8, and the sum of the exponents of every term amount to 8, and they will stand thus: a^8, a^7b, a^6b^2 a^5b^3, a^4b^4, a^3b^5, a^2b^6, ab^7, b^8.

The coefficients are not so obvious. However, we observe that the coefficients of the first and last terms must be *unity*. The coefficients of the terms next to the first and last are equal, and the same as the index of the power. The coefficients increase to the middle of the series, and then decrease in the same manner, and it is manifest that there must be *some* law of connection between the exponents and the coefficients.

By inspecting the 5th power of $a+b$, we find that the 2d coefficient is 5, and the 3d is 10.

$$\frac{5\times4}{2}=10.$$

The third coefficient is the 2d, multiplied by the exponent of the leading letter, and *divided by the exponent of the second letter increased by unity.*

In the same manner, the fourth coefficient is the third multiplied by the exponent of the leading letter, and divided by the exponent of the second letter *increased* by *unity*, and so on from coefficient to coefficient.

The 4th coefficient is $\frac{10\times3}{3}=10$

The 5th is $\frac{10\times2}{4}=5$

The last is $\frac{5\times1}{5}=1$ understood.

Now let us expand $(a+b)^8$

For the 1st term write a^8

For the 2d term write $8a^7b$

For the 3d, $\frac{8\times7}{2}=28$ $28a^6b^2$

For the 4th, $\frac{28\times6}{3}$ $56a^5b^3$

For the 5th, $\frac{56\times5}{4}$ $70a^4b^4$

Now as the exponents of a and b are equal, we have arrived at the middle of the power, and of course to the highest coefficient. The coefficients now decrease in the reverse order which they increased.

Hence the expanded power is

$$a^8+8a^7b+28a^6b^2+56a^5b^3+70a^4b^4+56a^3b^5+28a^2b^6+8ab^7+b^8.$$

Let the reader observe, that the exponent of b, increased by unity is always equal to the number of terms from the beginning, or from the left of the power. Thus, b^2 is in the 3d term, &c. Therefore in finding the coefficients we may divide by the number of terms already written, in place of the exponents of the second term increased by unity.

If the binomial $(a+b)$ becomes $(a+1,)$ that is, when b becomes unity, the 8th power becomes,

$$a^8+8a^7+28a^6+56a^5+70a^4+56a^3+28a^2+8a+1.$$

Any power of 1 is 1, and 1 as a factor never appears.

If a becomes 1, then the expanded power becomes,

$$1+8b+28b^2+56b^3+70b^4+56b^5+28b^6+8b^7+b^8$$

The manner of arriving at these results is to represent the unit by a letter, and *expand the simple literal terms*, and afterwards substitute their values in the result.

(Art. 68.) If we expand $(a-b)$ in place of $(a+b,)$ the exponents and coefficients will be precisely the same, but the principles of multiplication of quantities affected by different signs will give the *minus* sign to the second and to every alternate term.

Thus the 6th power of $(a-b)$ is

$$a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6.$$

(Art. 69.) This method of readily expanding the powers of a *binomial* quantity is one application of the "*binomial theorem*," and it was thus by induction and by observations on the result of particular cases that the theorem was established. Its rigid demonstration is somewhat difficult, but its application is simple and useful.

Its most general form may arise from expanding $(a+b)^n$.

When $n=3$, we can readily expand it;

When $n=4$, we can expand it;

When $n=$ any whole positive number, we can expand it.

Now let us operate with n just as we would with a known number, and we shall have

$$(a+b)^n=a^n+na^{n-1}b+n\frac{n-1}{2}a^{n-2}b^2, \quad \&c.$$

We know not where the series would terminate until we know the value of n. We are convinced of the truth of the result when n represents any positive whole number; but let n be negative or fractional, and we are not so sure of the result. To extend it to such cases requires deeper investigation and rigid demonstration, which it would not be proper to go into at this time. We shall, therefore, content ourselves with some of its more simple applications.

EXAMPLES.

1. Required the third power of $3x+2y$.

We cannot well expand this by the binomial theorem, because the terms are not simple *literal quantities*. But we can assume $3x=a$ and $2y=b$. Then

$$3x+2y=a+b \quad \text{and} \quad (a+b)^3=a^3+3a^2b+3ab^2+b^3.$$

[illegible] + 9x [illegible]

$4y^2$

[illegible] 36 [illegible]

Now to return to the values of a and b, we have,

$$\begin{array}{rl} a^3= & 27x^3 \\ 3a^2b=3\times 9x^2\times 2y= & 54x^2y. \\ 3ab^2=3\times 3x\times 4y^2= & 36xy^2. \\ b^3= & 8y^3. \end{array}$$

Hence $(3x+2y)^3=27x^3+54x^2y+36xy^2+8y^3$.

2. Required the 4th power of $2a^2-3$.

Let $x=2a^2$ $y=3$. Then expand $(x-y)^4$, and return the values of x and y, and we shall find the result,

$$16a^8-96a^6+216a^4-216a^2+81.$$

3. Required the cube of $(a+b+c+d)$.

As we can operate in this summary manner *only* on *binomial* quantities, we represent $a+b$ by x, or assume $x=a+b$, and $y=c+d$.

Then $(x+y)^3=x^3+3x^2y+3xy^2+y^3$.

Returning the values of x and y, we have

$$(a+b)^3+3(a+b)^2(c+d)+3(a+b)(c+d)^2+(c+d)^3.$$

Now we can expand by the binomial, these quantities contained in parenthesis.

4. Required the 4th power of $2a+3x$.

Ans. $16a^4+96a^3x+216a^2x^2+216ax^3+81x^4$.

5. Expand $(x^2+3y^2)^5$.

Ans. $x^{10}+15x^8y^2+90x^6y^4+270x^4y^6+405x^2y^8+243y^{10}$.

6. Expand $(2a^2+ax)^3$ *Ans* $8a^6+12a^5x+6a^4x^2+a^3x^3$.

7. Expand $(x-1)^6$.

Ans. $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.

8. Expand $(3x-5)^3$ *Ans.* $27x^3-135x^2+225x-125$.

9. Expand $(a+2)^4$. *Ans.* $a^4+8a^3+24a^2+32a+16$

10. Expand $(1-\frac{1}{2}a)^4$.. *Ans.* $1-2a+\frac{3a^2}{2}-\frac{a^3}{2}+\frac{a^4}{16}$

11. Expand $(a+b+c)^2$.

12. Expand $(a-2b)^3$.

13. Expand $(1-2x)^5$.

EVOLUTION.

CHAPTER II.

(Art. 70). Evolution is the converse of involution, or the extraction of roots, and the main principle is to *observe* how powers are formed, to be able to trace the operations back. Thus, to square a, we double its exponent, (Art. 65), and make it a^2. Square this and we have a^4. Cube a^2 and we have a^6. Take the 4th power of x and we have x^4. The nth power of x^2 is x^{2n}.

Now, if *multiplying* exponents raises simple literal quantities to powers, *dividing* exponents must extract roots. Thus, the square root of a^4 is a^2. The cube root of a^2 must be $a^{\frac{2}{3}}$. The cube root of a must have its exponent, (1 understood,) divided by 3, which will make $a^{\frac{1}{3}}$.

Therefore roots are properly expressed by fractional exponents.

The square root of a is $a^{\frac{1}{2}}$, and the exponents, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c. indicate the third, fourth, and fifth roots. The 6th root of x^5 is $x^{\frac{5}{6}}$; hence we perceive that the numerators of the exponent indicate the power of the quantity, and the denominator the root of that power.

(Art. 71.) The square of ax is a^2x^2. We square both factors, and so, for any other power, we raise *all* the factors to the required power. Conversely, then, we extract roots by taking the required root of all the factors. Thus the cube root of $8x^3$ is $2x$.

A root that can not be exactly expressed is called a *surd*, or *irrational* root. The square root of 2 can not be exactly expressed, hence, it is called a *surd*. A root merely expressed by the radical sign ($\sqrt{}$), or by a fractional exponent, is called a *radical* quantity. The cube root of 9 may be symbolically expressed thus, $\sqrt[3]{9}$, or $(9)^{\frac{1}{3}}$, and this is a radical quantity; but numerically the root can not be exactly expressed; it is, therefore, a *surd*. *Surd* roots may be found *approximately*.

The square root of $64a^6$ is obviously $8a^3$, and from this and the preceding examples we draw the following

RULE. *For the extraction of the roots of monomials. Extract the root of the numeral coefficients and divide the exponent of each letter by the index of the root.*

EXAMPLES.

1. What is the square root of $49a^2x^4$? *Ans.* $7ax^2$
2. What is the square root of $25c^{10}b^2$? *Ans.* $5c^5b$.
3. What is the square root of $20ax$? *Ans.* $2\sqrt{5ax}$.

In 20, the square factor 4 can be taken out; the other factor is 5. The square root of 4 is 2, which is all the root we can take; the root of the other factors can only be indicated as in the answer.

4. What is the square root of $12a^2$? *Ans.* $2a\sqrt{3}$.
5. What is the square root of $144a^2c^4x^2y^2$? *Ans.* $12ac^2xy$.
6. What is the square root of $36x^4$? *Ans.* $\pm 6x^2$.

(Art. 72.) The square root of algebraic quantities may be taken with the double sign, as indicating either *plus* or *minus*, for either quantity will give the same square, and we may not know which of them produced the power. For example, the square root of 16 may be either $+4$ or -4, for either of them, when multiplied by itself, will produce 16.

The cube root of a *plus* quantity is always plus, and the cube root of a *minus* quantity is always minus. For $+2a$ cubed gives $+8a^3$, and $-2a$ cubed gives $-8a^3$, and a may represent any quantity whatever.

EXAMPLES.

1. What is the cube root of $125a^3$? *Ans.* $5a$.
2. What is the cube root of $-64x^6$? *Ans.* $-4x^2$
3. What is the cube root of $-216a^3y^9$? *Ans.* $-6ay^3$
4. What is the cube root of $729a^6x^{12}$? *Ans.* $9a^2x^4$
5. What is the cube root of $32a^5$? *Ans.* $2a\sqrt[3]{4a^2}$

6. What is the 4th root of $256a^4x^8$? *Ans.* $\pm 4ax^2$.

7. What is the 4th root of $16a$? *Ans.* $\pm 2a^{\frac{1}{4}}$.

8. What is the 4th root of $64x^2y^2$? *Ans.* $\pm\sqrt{8xy}$.

N. B. The 4th root is the square root of the square root.

9. What is the 4th root of $20ax$? *Ans.* $\pm(20)^{\frac{1}{4}}a^{\frac{1}{4}}x^{\frac{1}{4}}$.

10. What is the square root of 75? *Ans.* $\pm 5\sqrt{3}$.

$$75=25\times 3.$$

11. Required the square root of $\frac{4a^2x^4}{9a^2}$. *Ans.* $\pm\frac{2x^2}{3}$.

12. Reqrired the square root of $\frac{32a^3x^5}{18ax}$. *Ans.* $\pm\frac{4ax^2}{3}$.

N. B. Reduce the fraction as much as possible, and then extract the root.

13. Required the square root of $\frac{200a^7}{128a}$. *Ans.* $\pm\frac{5a^3}{4}$.

14. Required the nth root of $\frac{a^nx^{2n}}{c^{2n}y^n}$. *Ans.* $\frac{ax^2}{c^2y}$.

15. Required the nth root of $\frac{a^3}{bc}$. *Ans.* $a^{\frac{3}{n}}b^{\frac{-1}{n}}c^{\frac{-1}{n}}$.

Observing that $\frac{1}{bc}=b^{-1}c^{-1}$

CHAPTER III.

To extract roots of compound quantities.

(Art. 72.) We shall commence this investigation by confining our attention to square root, and the only principle to guide us is the law of formation of squares. The square of $a+b$ is $a^2+2ab+b^2$. Now on the supposition that we do not know the root of this quantity to be $a+b$, we are to find it or extract it out of the square

$$a^2+2ab+b^2.$$

We know that a^2, the first term, must have been formed by the multiplication of a into itself, and the next term is $2a\times b$. That

$y = x +$ [illegible]

[illegible]

is *twice* the root of the first term into the second term of the root. Hence if we divide the second term of the square by twice the root of the first term, we shall obtain b, the second term of the root, and as b must be multiplied into itself to form a square, we add b to $2a$, and have $2a+b$, which we call a divisor.

OPERATION.

$$\begin{array}{ll} & a^2+2ab+b^2(a+b. \\ & a^2 \\ \hline 2a+b) & 2ab+b^2 \\ & 2ab+b^2 \\ \hline \end{array}$$

We take a for the first term of the root, and subtract its square (a^2) from the whole square. We then double a and divide $2ab$ by $2a$ and we find b, which we place in both the divisor and quotient. Then we multiply $2a+b$ by b and we have $2ab+b^2$, to subtract from the two remaining terms of the square, and in this case nothing remains.

Again, let us take $a+b+c$, and square it. We shall find its square to be

$$a^2+2ab+b^2+2ac+2bc+c^2.$$

$$\begin{array}{lll} & a^2+2ab+b^2+2ac+2bc+c^2(a+b+c & \\ & a^2 & \\ \hline 2a+b & \quad 2ab+b^2 & \\ & \quad 2ab+b^2 & \\ \hline 2a+2b+c & & 2ac+2bc+c^2 \\ & & 2ac+2bc+c^2 \\ \hline \end{array}$$

By operating as before, we find the first two terms of the root to be $a+b$, and a remainder of $2ac+2bc+c^2$. Double the root already found, and we have $2a+2b$ for a partial divisor. Divide the first term of the remainder $2ac$ by $2a$, and we have c for the third term of the root, which must be added to $2a+2b$ to complete the divisor. Multiply the divisor by the last term of the root and set the product under the three terms last brought down, and we have no remainder.

Again, let us take $a+b+c$ to square; but before we square it let the single letter $s=a+b$.

Then we shall have $s+c$ to square, which produces $s^2+2sc+c^2$. To take the square root of this we repeat the first operation, and thus the root of any quantity can be brought into a binomial and the rule for a binomial root will answer for a root containing any number of terms by considering the root *already found*, however great, as *one* term.

Hence the following *rule to extract the square root of a compound quantity.*

Arrange the terms according to the powers of some letter, beginning with the highest, and set the square root of the first term in the quotient.

Subtract the square of the root thus found from the first term and bring down the next two terms for a dividend.

Divide the first term of the dividend by double of the root already found, and set the result both in the root and in the divisor.

Multiply the divisor, thus completed, by the term of the root last found, and subtract the product from the dividend, and so on.

EXAMPLES.

1. What is the square root of

$$\begin{array}{lll}
 & a^4+4a^2b+4b^2-4a^2-8b+4\,(a^2+2b-2 & \\
 & a^4 & \\
\cline{2-2}
2a^2+2b &)\,4a^2b+4b^2 & \\
 & \;\;4a^2b+4b^2 & \\
\cline{2-3}
2a^2+4b-2 & & -4a^2-8b+4 \\
 & & -4a^2-8b+4 \\
\cline{3-3}
\end{array}$$

2. What is the square root of $1-4b+4b^2+2y-4by+y^2$?
Ans. $1-2b+y$.

3. What is the square root of $4x^4-4x^3+13x^2-6x+9$?
Ans. $2x^2-x+3$.

4. What is the square root of $4x^4-16x^3+24x^2-16x+4$?
Ans. $2x^2-4x+2$.

5. What is the square root of $16x^4+24x^3+89x^2+60x+100$?

Ans. $4x^2+3x+10$.

6 What is the square root of $4x^4-16x^3+8x^2+16x+4$?

Ans. $2x^2-4x-2$

7. What is the square root of

$x^2+2xy+y^2+6xz+6yz+9z^2$? *Ans.* $x+y+3z$

8. What is the square root of $a^2-ab+\frac{1}{4}b^2$?

Ans. $a-\frac{1}{2}b$.

9. What is the square root of $\frac{a^2}{b^2}-2+\frac{b^2}{a^2}$?

Ans. $\frac{a}{b}-\frac{b}{a}$ or $\frac{b}{a}-\frac{a}{b}$

10. What is the square root of $x^{\frac{2}{3}}-2x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$?

Ans. $x^{\frac{1}{3}}-y^{\frac{1}{3}}$ or $y^{\frac{1}{3}}-x^{\frac{1}{3}}$.

(Art. 73.) Every square root will be equally a root if we change the sign of all the terms. In the first example, for instance, the root may be taken $-a^2-2b+2$, as well as a^2+2b-2, for either one of these quantities, by squaring, will produce the given square. Also, observe that every square consisting of three terms only, has a binomial root.

(Art. 74.) Algebraic squares may be taken for formulas, corresponding to numeral squares, and their roots may be extracted in the same way, and by the *same rule.*

For example, $a+b$ squared is $a^2+2ab+b^2$, and to apply this to numerals, suppose $a=40$ and $b=7$.

Then the square of 40 is $a^2=1600$

$2ab=\ \ 560$

$b^2=\ \ \ \ 49$

Therefore, $(47)^2=2209$

Now the necessary divisions of this square number, 2209, are not visible, and the chief difficulty in discovering the root is to make these separations.

The first observation to make is that the square of 10 is 100, of 100 is 10000, and so on. Hence, the square root of any square number less than 100 consists of one figure, and of any square number over 100 and less than 10000 of two figures, and so on. *Every two places in a power demanding one place in its root.*

Hence, to find the number of places or figures in a root, we must *separate the power into periods of two figures, beginning at the unit's place.* For example, let us require the square root of 22·09. Here are two periods indicating two places in the root, corresponding to tens and units. The greatest square in 22 is 16, its root is 4, or 4 tens $=40$. Hence $a=40$.

$$\begin{array}{rl} & 22\ 09(40+7=47 \\ a^2= & \underline{16\ 00} \\ 2a+b=80+7=87 &)609 \\ & \underline{609} \end{array}$$

Then $2a=80$, which we use as a divisor for 609, and find it is contained 7 times. The 7 is taken as the value of b, and $2a+b$, the complete divisor, is 87, which multiplied by 7 gives the two last terms of the binomial square. $2ab+b^2=560+49=609$, and the entire root $40+7=47$ is found.

Arithmetically, a may be taken as 4 in place of 40, and 1600 as 16, the place occupied by the 16 makes it 16 hundred, and the ciphers are superfluous. Also, $2a$ may be considered 8 in place of 80, and 8 in 60 (not in 609) is contained 7 times, &c.

If the square consists of more than *two periods*, treat it as *two*, and obtain the two superior figures of the root, and when obtained bring down another period to the remainder, and consider the root already obtained as one quantity, or one figure.

For another example, let the square root of 399424 be extracted.

$$\begin{array}{r|l} & 39\cdot94\cdot24 \ \| \ 632 \\ & \underline{36} \\ 123 & 394 \\ & \underline{369} \\ 1262 & 25\ 24 \\ & 25\ 24 \end{array}$$

In this example, if we disregard the local value of the figures, we have $a=6$, $2a=12$, and 12 in 39, 3 times, which gives $b=3$. Afterwards we suppose $a=63$, and $2a=126$, 126 in 252, 2 times, or the second value of $b=2$. In the same manner, we would repeat the formula of a binomial square as many times as we have periods.

EXERCISES FOR PRACTICE.

1. What is the square root of 8836? *Ans.* 94.

2. What is the square root of 106929? *Ans.* 327.

3. What is the square root of 4782969? *Ans.* 2187.

4. What is the square root of 43046721? *Ans.* 6561.

5. What is the square root of 387420489? *Ans.* 19683.

When there are whole numbers and decimals, point off periods both ways from the decimal point, and make the decimal places even, by annexing ciphers when necessary, extending the decimal as far as desired. When there are decimals only, commence pointing off from the decimal point.

EXAMPLES.

1. What is the square root of 10·4976? *Ans.* 3·24.

2. What is the square root of 3271·4207? *Ans.* 57·19+.

3. What is the square root of 4795·25731? *Ans.* 69·247+.

4. What is the square root of ·0036? *Ans.* ·06

5. What is the square root of ·00032754? *Ans.* ·01809+.

6. What is the square root of ·00103041? *Ans.* ·0321.

(Art. 75.) As the square of any quantity is the quantity multiplied by itself, and the product of $\frac{a}{b}$ by $\frac{a}{b}$ (Art. 64.) is $\frac{a^2}{b^2}$; hence *to take the square root of a fraction we must extract the square root of both numerator and denominator.*

A fraction may be *equal* to a square, and the terms, as given, not square numbers; such may be reduced to square numbers.

EXAMPLES.

What is the square root of $\frac{72}{128}$?

Observe $\frac{72}{128}=\frac{36}{64}$. Hence the square root is $\frac{6}{8}$.

1. What is the square root of $\frac{98}{128}$? *Ans.* $\frac{7}{8}$.
2. What is the square root of $\frac{112}{175}$? *Ans.* $\frac{4}{5}$.
3. What is the square root of $\frac{2304}{5184}$? *Ans.* $\frac{2}{3}$.
4. What is the square root of $\frac{2704}{4225}$? *Ans.* $\frac{4}{5}$.

When the given fractions cannot be reduced to square terms, reduce the value to a decimal, and extract the root, as in the last article.

CHAPTER IV.

To extract the cube root of compound quantities.

(Art. 76.) We may extract the cube root in a similar manner as the square root, by dissecting or retracing the combination of terms in the formation of a binomial cube.

The cube of $a+b$ is $a^3+3a^2b+3ab^2+b^3$ (Art. 67). Now to extract the root, it is evident we must take the root of the first term (a^3), and the next term is $3a^2b$. *Three times the square of the first letter or term of the root multiplied by the 2d term of the root.*

Therefore to find this second term of the root we must divide the second term of the power ($3a^2b$) by three times the square of the root already found (a).

$$\begin{array}{l} 3a^2)3a^2b(b \\ \underline{3a^2b} \end{array}$$

When we can decide the value of b, we may obtain the complete divisor for the remainder after the cube of the first term is subtracted, thus:

The remainder is $3a^2b+3ab^2+b^3$

Take out the factor b, and $3a^2+3ab+b^2$ is the complete divisor for the remainder. But this divisor contains b, the very term we wish to find by means of the divisor; hence it must be found before the divisor can be completed. In distinct algebraic quantities there can be no difficulty, as the terms stand separate, and we find b by dividing simply $3a^2b$ by $3a^2$; but in numbers the terms are mingled together, and b can only be found by trial.

Again, the terms $3a^2+3ab+b^2$ explain the common arithmetical rule, as $3a^2$ stands in the place of hundreds, it corresponds with the words: "Multiply the square of the quotient by 300," "and the quotient by 30," ($3a$,) &c.

By inspecting the various powers of $a+b$, (Art. 67,) we draw the following general rule for the extraction of roots:

Arrange the terms according to the powers of some letter; take the required root of the first term and place it in the quotient: subtract its corresponding power from the first term, and bring down the second term for a dividend.

Divide this term by *twice* the root already *found for the* SQUARE *root, three times the square of it for the* CUBE *root, four times the third power for the fourth root, &c., and the quotient will be the next term of the root. Involve the whole of the root, thus found, to its proper power, which subtract from the given quantity, and divide the first term of the remainder by the same divisor as before: proceed in this manner till the whole root is determined.*

EXAMPLES.

1. What is the cube root of $x^6+6x^5-40x^3+96x-64$?

$x^6+6x^5-40x^3+96x-64$ $(x^2+2x-4$

x^6

Divisor $3x^4)$ $6x^5$=1st remainder.

$x^6+6x^5+12x^4+8x^3=(x^2+2x)^3$

Divisor $3x^4$ $)$ $-12x^4$=2d remainder.

$x^6+6x^5-40x^3+96x-64$

2. What is the cube root of $27a^3+108a^2+144a+64$?

Ans. $3a+4$.

3. What is the cube root of $a^3-6a^2x+12ax^2-8x^3$?

Ans. $a-2x$

4. What is the cube root of $x^6-3x^5+5x^3-3x-1$?

Ans. x^2-x-1

5. What is the cube root of $a^3-6a^2b+12ab^2-8b^3$?

Ans. $a-2b$.

6. What is the cube root of $x^3+3x+\frac{3}{x}+\frac{1}{x^3}$?

Ans. $x+\frac{1}{x}$.

7. Extract the fourth root of

$$\begin{array}{ll}
 & a^4+8a^3+24a^2+32a+16(a+2 \\
 & a^4 \\
\hline
4a^3) & 8a^3,\quad \&c. \\
\hline
 & a^4+8a^3+24a^2+32a+16. \\
\hline
\end{array}$$

(Art. 77.) To apply this general rule to the extraction of the cube root of numbers, we must first observe that the cube of 10 is 1000, of 100 is 1000000, &c.; ten times the root producing 1000 times the power, or one cipher in the root producing 3 in the power; hence any cube within 3 places of figures can have only one in its root, any cube within 6 places can have only two places in its root, &c. Therefore we must divide off the given power into periods consisting of three places, commencing at the unit. If the power contains decimals, commence at the unit place, and count three places each way, and the number of periods will indicate the number of figures in the root.

EXAMPLES.

1. Required the cube root of 12812904.

$$\begin{array}{lll}
 & & 12{\cdot}812{\cdot}904(234 \\
a=2 & a^3= & 8 \\
\hline
\text{Divisor} & 3a^2=12 &)48 \\
\hline
 & & 12167=(23)^3 \\
\hline
 & 3(23)^2=1587) & 6459\ (4 \\
\hline
 & & 12\,812\,904=(234)^3 \\
\hline
\end{array}$$

Here 12 is contained in 48, 4 times; but it must be remembered that 12 is only a trial or partial divisor; when completed it will exceed 12, and of course the next figure of the root cannot exceed 3.

The first figure in the root was 2. Then we assumed $a=2$. Afterwards we found the next figure must be 3. Then we assumed $a=23$. To have found a succeeding figure, had there been a remainder, we should have assumed $a=234$, &c., and from it obtained a new partial divisor.

2. What is the cube root of 148877? *Ans.* 53.

3. What is the cube root of 571787? *Ans.* 83.

4. What is the cube root of 1367631? *Ans.* 111.

5. What is the cube root of 2048383? *Ans.* 127.

6. What is the cube root of 16581375? *Ans.* 255.

7. What is the cube root of 44361864? *Ans.* 354.

8. What is the cube root of 100544625? *Ans.* 465.

(Art. 78.) When the power is not complete, and, of course, its root surd, the methods of direct extraction are all too tedious to be much used, and several eminent mathematicians have given more brief and practical methods of approximation.

One of the most useful methods may be investigated as follows:

Suppose a and $a+c$ two cube roots, c being *very small* in relation to a. a^3 and $a^3+3a^2c+3ac^2+c^3$ are the cubes of the supposed roots.

Now if we double the first cube (a^3), and add it to the second, we shall have

$$3a^3+3a^2c+3ac^2+c^3.$$

If we double the second cube and add it to the first, we shall have

$$3a^3+6a^2c+6ac^2+2c^3.$$

As c is a very small fraction compared to a, the terms containing c^2 and c^3 are very small in relation to the others, and the relation of these two sums will not be materially changed by rejecting those terms containing c^2 and c^3, and the same will then be

$$3a^3+3a^2c$$

And $$3a^3+6a^2c$$

The ratio of these terms is the same as the ratio of $a+c$ to $a+2c$.

Or the ratio is $$1+\frac{c}{a+c}$$

But the ratio of the roots a to $a+c$, is $1+\frac{c}{a}$.

Observing again that c is supposed to be very small in relation to a, the fractional parts of the ratios $\frac{c}{a+c}$ and $\frac{c}{a}$ are both small and very near in value to each other. Hence we have found an operation on two cubes which are near each other in magnitude, that will give a proportion very *near* in proportion to their roots; and by knowing the root of one of the cubes, by this ratio we can find the other.

For example, let it be required to find the cube root of 28, true to 4 or 5 places of decimals. As we wish to find the cube root of 28, we may assume that 28 is a *cube*. 27 is a cube near in value to 28, and the root of 27 we know to be 3.

Hence a, in our investigation, corresponds to 3 in this example, and c is unknown; but the cube of $a+c$ is 28, and a^3 is 27.

$$\begin{array}{lrcrcccl}
\text{Then} & 27 & & 28 & & & & \\
 & 2 & & 2 & & & & \\
\hline
 & 54 & & 56 & & & & \\
\text{Add} & 28 & & 27 & & & & \\
\hline
\text{Sums} & 82 & : & 83 & :: & 3 & : & a+c \quad \text{very nearly.}
\end{array}$$

Or $(a+c)=\frac{249}{82}=3{\cdot}03658+$, which is the cube root of 28, true to 5 places of decimals.

By the laws of proportion, which we hope more fully to investigate in a subsequent part of this work, the above proportion, $82 : 83 :: a : a+c$, may take this change

$$82 : 1 :: a : c$$

Hence, $c=\frac{3}{82}$. c being a correction to the known root, which, being applied, will give the unknown or sought root.

From what precedes, we may draw the following rule for finding approximate cube roots:

RULE. *Take the nearest rational cube to the given number, and call it an assumed cube; or, assume a root to the given number and cube it. Double the assumed cube and add the given number to it; also, double the given number and add the assumed cube to it. Then, by proportion, as the first sum is to the second, so is the known root to the required root. Or take the difference of these sums, then say, as double of the assumed cube, added to the number, is to this difference, so is the assumed root to a correction.*

This correction, added to or subtracted from the assumed root, *as the case may require*, will give the cube root very nearly.

By repeating the operation with the root last found as an assumed root, we may obtain results to any degree of exactness; one operation, however, is generally sufficient.

EXAMPLES.

1. What is the approximate cube root of 120?

Ans. 4·93242+.

2. What is the approximate cube root of 8·5?

Ans. 2·0408+.

3. What is the approximate cube root of 63?

Ans. 3·979057+.

4. What is the approximate cube root of 515?

Ans. 8·01559+.

5. What is the approximate cube root of 16?

The cube root of 8 is 2, and of 27 is 3; therefore the cube root of 16 is between 2 and 3. Suppose it 2·5. The cube of this root is 15·625, which shows that the cube root of 16 is a little more than 2·5, and by the rule

31·25		32				
16		15·625				
47·25	:	47·625	::	2·5	:	to the required root.
47·25	:	·375	::	2·5	:	·01984

Assumed root	2·50000
Correction	·01984
Approximate root	2·51984.

We give the last as an example to be followed in most cases where the root is about midway between two integer numbers.

This rule may be used with advantage to extract the root of perfect cubes, when the powers are very large.

EXAMPLE.

The number 22·069·810·125 is a cube; required its root.

Dividing this cube into periods, we find that the root must contain 4 figures, and the superior period is 22, and the cube root of 22 is near 3, and of course the whole root near 3000; but less than 3000. Suppose it 2800 and cube this number. The cube is 21952000000, which being less than the given number, shows that our assumed root is not large enough.

To apply the rule, it will be sufficient to take six superior figures of the given and assumed cubes. Then by the rule,

$$\begin{array}{rcrcr}
219\,520 & & 220698 & & \\
2 & & 2 & & \\
\cline{1-1}\cline{3-3}
439040 & & 441396 & & \\
220698 & & 219520 & & \\
\cline{1-1}\cline{3-3}
659738 & : & 660916 & :: & 2800 \\
 & & 659738 & & \\
\cline{3-3}
659738 & : & 1178 & :: & 2800 \\
 & & 2800 & & \\
\cline{3-3}
 & & 942400 & & \\
 & & 2356 & & \\
\cline{3-3}
 & 659738) & 3298400(5 & & \\
 & & 3298690 & & \\
\cline{3-3}
\end{array}$$

Assumed root, 2800
Correction, 5
True root, 2805

The result of the last proportion is not exactly 5, as will be seen by inspecting the work; the slight imperfection arises from the rule being approximate, not perfect.

When we have cubes, however, we can always decide the unit figure by inspection, and, in the present example, the unit figure in the cube being 5, the unit figure in the root must be 5, as no other figure when cubed will give 5 in the place of units.

[For several other abbreviations and expedients in extracting cube root in numerals, see Robinson's Arithmetic.]

(Art. 79.) To obtain the 4th root, we may extract the square root of the square root. To obtain the 6th root, we may take the square root first, and then the cube root of that quantity.

To extract odd roots of high powers in numeral quantities is very tedious and of no practical utility; we therefore give no examples.

(Art. 80.) Some radical quantities may be simplified, and others are not susceptible of simplification. For instance, the square root of 75, written $\sqrt{75}$, is equal to 5 times the square root of 3; that is, $\sqrt{75}=5\sqrt{3}$. But the square root of 71 can not be fully expressed except by the sign, thus, $\sqrt{71}$, because 71 contains no *square factor*.

It is obvious that $\sqrt[3]{40}=\sqrt[3]{8\cdot5}=\sqrt[3]{8}\ \sqrt[3]{5}=2(5)^{\frac{1}{3}}$; but 39 and 41 containing no *cube factor*, their cube roots can only be indicated by the radical sign over them.

When simplification is possible, it can be effected by the following

Rule. *Separate the quantity into two factors, one of which is a perfect power of the required root. Extract the root of that factor and prefix the result as a coefficient to the other factor placed under the radical sign.*

We give the following examples for practice:

1. Reduce the square root of 75 to lower terms, or reduce $\sqrt{75}$. *Ans.* $5\sqrt{3}$.
2. Reduce $\sqrt{98a^2}$ to lower terms. *Ans.* $7a\sqrt{2}$.
3. Reduce $\sqrt{12x^2y}$ to lower terms. *Ans.* $2x\sqrt{3y}$.
4. Reduce $\sqrt[3]{54x^4}$ to lower terms. *Ans.* $3x\sqrt[3]{2x}$.
5. Reduce $4\sqrt[3]{108}$ to lower terms. *Ans.* $12\sqrt[3]{4}$.
6. Reduce $\sqrt{x^3-a^2x^2}$ to lower terms. *Ans.* $x\sqrt{x-a^2}$.

7. Reduce $\sqrt[3]{32a^3}$ to lower terms. *Ans.* $2a\sqrt[3]{4}$.

8. Reduce $\sqrt{28a^3x^2}$ to lower terms. *Ans.* $2ax\sqrt{7a}$

9. Reduce $\sqrt{\frac{44}{75}}$ to lower terms.

Where terms under the radical are fractional, it is expedient to reduce the denominator to a power corresponding to the radical sign; then by extracting the root there will be no fraction under the radical.

The above example may be treated thus:

$$\sqrt{\tfrac{44}{75}}=\sqrt{\tfrac{4}{25}\times\tfrac{11}{3}}=\sqrt{\tfrac{4}{25}\times\tfrac{33}{9}}=\sqrt{\tfrac{4}{25\cdot 9}\times 33}=\tfrac{2}{15}\sqrt{33}. \quad \textit{Ans.}$$

We divided $\frac{44}{75}$ into the factors $\frac{4}{25}$ and $\frac{11}{3}$; the first factor is a *square;* the other factor, $\frac{11}{3}$, we multiply both numerator and denominator by 3, to make the denominator a square.

In like manner reduce the following:

10. Reduce $\sqrt[3]{\frac{135}{32}}$ to more simple terms. *Ans.* $\frac{3}{4}\sqrt[3]{10}$.

11. Reduce $\sqrt[3]{\frac{25}{9}}$ to more simple terms. *Ans.* $\frac{1}{3}\sqrt[3]{75}$.

12. Reduce $\sqrt{\frac{50}{147}}$ to more simple terms. *Ans.* $\frac{5}{21}\sqrt{6}$.

13. Reduce $\sqrt[3]{a^3+a^3b^2}$ to more simple terms.

Ans. $a\sqrt[3]{1+b^2}$.

14. Reduce $\sqrt{\frac{2a}{3}}$ to more simple terms. *Ans.* $\frac{1}{3}\sqrt{6a}$.

(Art. 81.) Radical quantities may be put into one sum, or the difference of two may be determined, provided the parts essentially radical are the same.

Thus the sum of $\sqrt{8}$ and $\sqrt{72}$ is $8\sqrt{2}$ and their difference is $4\sqrt{2}$

$$\begin{array}{ll}
\text{For} & \sqrt{8}=2\sqrt{2} \\
\text{And} & \sqrt{72}=6\sqrt{2} \\
\hline
\text{Sum} & \quad\;\; 8\sqrt{2} \\
\text{Difference} & \quad\;\; 4\sqrt{2}
\end{array}$$

When radical quantities are not and cannot be reduced to the same quantity under the sign, their sum and difference can only be taken by the signs plus and minus.

EXAMPLES.

1. Find the sum and difference of $\sqrt{16a^2x}$ and $\sqrt{4a^2x}$.

Ans. Sum, $6a\sqrt{x}$; difference, $2a\sqrt{x}$.

2. Find the sum and difference of $\sqrt{128}$ and $\sqrt{72}$.

Ans. Sum, $14\sqrt{2}$; difference, $2\sqrt{2}$.

3. Find the sum and difference of $\sqrt[3]{135}$ and $\sqrt[3]{40}$.

Ans. Sum, $5\sqrt[3]{5}$; difference, $\sqrt[3]{5}$.

4. Find the sum and difference of $\sqrt[3]{108}$ and $9\sqrt[3]{4}$.

Ans. Sum, $12\sqrt[3]{4}$; difference, $6\sqrt[3]{4}$.

5. Find the sum and difference of $\sqrt{\frac{2}{4}}$ and $\sqrt{\frac{2}{9}}$.

Ans. Sum, $\frac{5}{6}\sqrt{2}$; difference, $\frac{1}{6}\sqrt{2}$.

6. Find the sum and difference of $\sqrt[3]{56}$ and $\sqrt[3]{189}$.

Ans. Sum, $5\sqrt[3]{7}$; difference, $\sqrt[3]{7}$.

7. Find the sum and difference of $3\sqrt{a^2b}$ and $3\sqrt{16a^4b}$.

Ans. Sum, $(12a^2+3a)\sqrt{b}$; difference, $(12a^2-3a)\sqrt{b}$.

(Art. 82.) We multiply letters together by writing them one after another, as *abxy*. If they are numeral quantities, their product appears as a number; if two or more of them are numeral, the product of these quantities will appear as a number.

This fundamental principle of multiplication may be applied to the multiplication of surds. Let it be required to multiply $5\sqrt{2}$ by $3\sqrt{7}$. Here suppose $a=5$, $b=3$, $x=\sqrt{2}$, $y=\sqrt{7}$. Then the product of $5\sqrt{2}$ by $3\sqrt{7}$ is $abxy$ or $15\sqrt{2\times 7}=15\sqrt{14}$. Hence, for the multiplication of quantities affected by the same radical sign, we draw the following

Rule. *Multiply the rational parts together for the rational part of the product, and the radical parts together for the radical part of the product.*

EXAMPLES.

1. Required the product of $5\sqrt{5}$ and $3\sqrt{8}$.

Product reduced. *Ans.* $30\sqrt{10}$.

2. Required the product of $4\sqrt{12}$ and $3\sqrt{2}$. *Ans.* $24\sqrt{6}$.

3. Required the product of $3\sqrt{2}$ and $2\sqrt{8}$. *Ans.* 24.

4. Required the product of $2\sqrt[3]{14}$ and $3\sqrt[3]{4}$.

Ans. $12\sqrt[3]{7}$.

5. Required the product of $2\sqrt{5}$ and $2\sqrt{10}$. *Ans.* $20\sqrt{2}$.

(Art. 83.) We multiply different powers of the same quantity together by simply adding the exponents as, a^3 multiplied by a^5 is a^8, and this holds when the exponents are fractional, as $a^{\frac{1}{2}}$ into $a^{\frac{1}{3}}$; for the product we simply add $\frac{1}{2}$ and $\frac{1}{3}$, which make $\frac{5}{6}$ to write over a for the product of $a^{\frac{1}{2}}$ into $a^{\frac{1}{3}}$ which gives $a^{\frac{5}{6}}$.

When the radical quantities are different, as $(a)^{\frac{1}{2}}$ and $b^{\frac{1}{3}}$, and the exponents different, are to be multiplied together, or one to be divided by another, we better lay aside all rules and apply that *powerful engine*, an equation. For illustration:

Required the product of $a^{\frac{1}{2}}$ into $b^{\frac{1}{3}}$.

The product is P, then we have

$$P=a^{\frac{1}{2}}b^{\frac{1}{3}}.$$

Raise both members to the 6th power, then,

$$P^6=a^3b^2.$$

Conceiving the second member to be a single quantity, and taking the 6th root, will give us

$$P=(a^3b^2)^{\frac{1}{6}}.$$

EXAMPLES.

1. Required the product of $(a+b)^{\frac{1}{3}}$ and $(a+b)^{\frac{3}{4}}$.

Ans. $(a+b)^{\frac{13}{12}}$.

2. Required the product of $\sqrt{7}$ and $\sqrt[3]{7}$. *Ans.* $(7^5)^{\frac{1}{6}}$

(Art. 84.) If we divide one quantity by another, the product of the quotient and the divisor must equal the dividend.

Divide the $\sqrt[6]{72}$ by $\sqrt{2}$: the quotient is Q.

Then, $(2)^{\frac{1}{2}}Q=(72)^{\frac{1}{6}}$.

By squaring, $2Q^2=(72)^{\frac{1}{3}}$.

Cubing $8Q^6=72$.

Whence, $Q^6=9 \quad Q^2=3 \quad Q=(3)^{\frac{1}{2}}$.

3. Required the product of $2\sqrt{3}$ and $3(4)^{\frac{1}{3}}$.

Ans. $6(432)^{\frac{1}{6}}$.

4. Required the product of $\sqrt[3]{15}$ and $\sqrt{10}$.

Ans. $\sqrt[6]{225000}$.

EXAMPLES.

1. Divide $4\sqrt{50}$ by $2\sqrt{5}$. *Ans.* $2\sqrt{10}$.

2. Divide $6\sqrt[3]{100}$ by $3\sqrt[3]{5}$. *Ans.* $2\sqrt[3]{20}$.

3. Divide $\sqrt{7}$ by $\sqrt[3]{7}$.

Let $a=7$. Then the example is, to divide the $a^{\frac{1}{2}}$ by $a^{\frac{1}{3}}$, or $a^{\frac{3}{6}}$ by $a^{\frac{2}{6}}$; quotient $a^{\frac{1}{6}}$. As we always subtract the exponents of like quantities to perform division (Art. 17); therefore $7^{\frac{1}{6}}$ must be the required quotient.

4. Divide $6\sqrt{54}$ by $3\sqrt{2}$. *Ans.* $6\sqrt{3}$.

5. Divide $(a^2b^2d^3)^{\frac{1}{6}}$ by $d^{\frac{1}{2}}$. *Ans.* $(ab)^{\frac{1}{3}}$.

6. Divide $(16a^3-12a^2x)^{\frac{1}{2}}$ by $2a$. *Ans.* $(4a-3x)^{\frac{1}{2}}$.

(Art. 85.) In the course of algebraical investigations, we might fall on the square root of a minus quantity, as $\sqrt{-a}$, $\sqrt{-1}$, $\sqrt{-b}$, &c., and it is important that the pupil should readily understand that such quantities have no real existence; for no quantity, either plus or minus, multiplied by itself will give minus a, minus b, or minus any quantity whatever; hence there is no value to $\sqrt{-a}$, &c., and such symbols are said to be impossible or *imaginary*.

(Art. 86.) The square and cube root of any quantity, as a, being expressed by $\sqrt{a}$ and $\sqrt[3]{a}$, and as by involving the root we obtain the power, hence the square of $\sqrt{a}$, is a; and the cube of $\sqrt[3]{a}$, is a. Hence, *removing the sign involves to the corresponding power.*

EXAMPLES.

1. What is the square of $\sqrt{3ax}$? *Ans.* $3ax$.

2. What is the cube of $\sqrt[3]{6y^2}$? *Ans.* $6y^2$.

3. What is the square of $\sqrt{a^2-x^2}$? *Ans.* a^2-x^2.

4. What is the square of $\sqrt{\dfrac{3a}{6+x}}$? *Ans.* $\dfrac{3a}{6+x}$

5. What is the cube of $\sqrt[3]{1+a}$? *Ans.* $1+a$.

(Art. 87.) When we have two or more quantities, and the radical sign *not extending over the whole,* involution will not remove the radical, but will change it from one term to another. Thus, $\sqrt{x}+a$ the square will be $x+2a\sqrt{x}+a^2$; the radical sign is still present, but not in the first term.

* * *

PURE EQUATIONS.

CHAPTER IV.

(Art. 88.) Pure equations in general are those in which a complete power of the unknown quantity is contained, and no special artifice is requisite to render the power complete.

The unknown quantity may appear in one or in several terms; when it appears in several, its exponents will be regular, descending from a higher to a lower value, or the reverse.

In such cases we must reduce the equation by *evolution.*

Like roots of equal quantities are equal. (Ax. 8.)

EXAMPLES.

1. Given $3x^2-9=66$ to find the value of x.

Solution, $3x^2=75$ $x^2=25$. Hence, $x=\pm5$.

We place the double sign before 5, as we cannot determine whether 25 was produced by the square of $+5$ or of -5.

In practical problems, the nature of the case will commonly determine, but in every abstract problem we must take the double sign.

2. Given $x^2-6x+9=a^2$ to find the value of x.

By evolution, $x-3=\pm a$. Hence, $x=3\pm a$, *Ans.*

3. Given $x^3-x^2+\frac{1}{3}x-\frac{1}{27}=a^3b^3$ to find x.

Take cube root and $x-\frac{1}{3}=ab$; and $x=ab+\frac{1}{3}$, *Ans.*

4. Given $x^4-2x^2+1=16$ to find the value of x.

Ans. $x=\pm\sqrt{5}$, or $\pm\sqrt{-3}$.

5. Given $x^2-4x+4=64$ to find the value of x.

Ans. $x=10$, or -6.

6. Given $x^2y^2+2xy+1=4x^2y^2$ and $x=2y$ to find the values of x and y. *Ans.* $x=\pm\sqrt{2}$, $y=\pm\frac{1}{2}\sqrt{2}$

7. Given $x^2+y^2=13$ and $x^2-y^2=5$ to find the values of x and y. *Ans.* $x=\pm 3$ $y=\pm 2$.

(Art. 89.) The unknown quantity of an equation is as likely to appear under a radical sign as to be involved to a power. In such cases we free the unknown quantity from radicals by involution (Art. 86.), having previously transposed all the terms not under the radical to one side of the equation, the radical being on the other.

9. Given $\sqrt{x+1}=a-1$ to find the value of x.

By involution, $x+1=a^2-2a+1$. Hence, $x=a^2-2a$.

10. Given $\sqrt{12+x}=3+\sqrt{x}$ to find the value of x.

Square both sides, and we have

$$12+x=9+6\sqrt{x}+x.$$

Drop $9+x$ and $3=6\sqrt{x}$; or $\sqrt{x}=\frac{1}{2}$. $x=\frac{1}{4}$, *Ans.*

11. Given $\sqrt{x-16}=8-\sqrt{x}$ to find x. *Ans.* $x=25$.

12. Given $\frac{x-ax}{\sqrt{x}}=\frac{\sqrt{x}}{x}$ to find x.

Multiply by $\sqrt{x}$, observing that x divided by x gives 1; and we have $x-ax=1$,

Or $(1-a)x=1$,

Therefore $x=\frac{1}{1-a}$.

13. Given $\frac{\sqrt{x}+28}{\sqrt{x}+4}=\frac{\sqrt{x}+38}{\sqrt{x}+6}$ to find the value of x.

By clearing of fractions, we have

$$x+34\sqrt{x}+168=x+42\sqrt{x}+152.$$

Reducing, $16=8\sqrt{x}$

By division, $2=\sqrt{x}$, or $4=x$.

To call out attention and cultivate tact, we give another solution. Divide each numerator by its denominator, and we have

$$1+\frac{24}{\sqrt{x}+4}=1+\frac{32}{\sqrt{x}+6}$$

Drop unity from both sides, and divide by 8; we then have

$$\frac{3}{\sqrt{x}+4}=\frac{4}{\sqrt{x}+6}$$

Clearing of fractions, $3\sqrt{x}+18=4\sqrt{x}+16$

Dropping equals, $2=\sqrt{x}$. Hence, $x=4$.

14. Given $\sqrt{x}+\sqrt{a+x}=\frac{2a}{\sqrt{a+x}}$ to find x. *Ans.* $x=\frac{1}{3}a$.

15. Given $x+\sqrt{a^2+x^2}=\frac{2a^2}{\sqrt{a^2+x^2}}$ to find x.

Ans. $x=a\sqrt{\frac{1}{3}}$

16. Given $x+a=\sqrt{a^2+x\sqrt{b^2+x^2}}$ to find x.

Ans. $x=\frac{b^2-4a^2}{4a}$.

17. Given $\frac{\sqrt{6x}-2}{\sqrt{6x}+2}=\frac{4\sqrt{6x}-9}{4\sqrt{6x}+6}$ to find x.* *Ans.* $x=6$.

18. Given $\sqrt[3]{64+x^2-8x}=\frac{4+x}{\sqrt[3]{4+x}}$ to find x. *Ans.* $x=3$

19. Given $\sqrt{5+x}+\sqrt{x}=\frac{15}{\sqrt{5+x}}$ to find x. *Ans.* $x=4$

20. Given $\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}=\frac{3}{2}\left(\frac{x}{x+\sqrt{x}}\right)^{\frac{1}{2}}$ to find x.

Ans. $x=\frac{25}{16}$.

21. Given $\frac{5x-9}{\sqrt{5x}+3}=1+\frac{\sqrt{5x}-3}{2}$ to find x.

Because $a^2-b^2=(a+b)(a-b)$

We infer that $5x-9=(\sqrt{5x}+3)(\sqrt{5x}-3)$

Therefore, $\frac{5x-9}{\sqrt{5x}+3}=\sqrt{5x}-3$. The given equation then becomes $\sqrt{5x}-3=1+\frac{\sqrt{5x}-3}{2}$.

Now assume $\sqrt{5x}-3=y$. (*A*)

Then $y=1+\frac{1}{2}y$. Consequently, $y=2$. Returning to equation (*A*), we have $\sqrt{5x}-3=2$

$\sqrt{5x}=5$. Therefore, $x=5$, *Ans.*

22. Given $\frac{\sqrt{ax}-b}{\sqrt{ax}+b}=\frac{3\sqrt{ax}-2b}{3\sqrt{ax}+5b}$ to find x. *Ans.* $x=\frac{9b^2}{a}$.

(Compare 22 with examples 13, and 17.)

23. Given $(1+x\sqrt{x^2+12})^{\frac{1}{2}}=1+x$ to find x. *Ans.* $x=2$.

24. Given $\frac{\sqrt{4x+1}+\sqrt{4x}}{\sqrt{4x+1}-\sqrt{4x}}=9$, to find x. *Ans.* $x=\frac{4}{9}$.

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* See 2d solution to Equation 13.

25. Given $a^2-2ax+x^2=b$, to find x. *Ans.* $x=a-\sqrt{b}$

26. Given $\dfrac{1}{1-\sqrt{1-x^2}}-\dfrac{1}{1+\sqrt{1-x^2}}=\dfrac{\sqrt{3}}{x^2}$ to find x.

Ans. $x=\pm\frac{1}{2}$

27. Given $\dfrac{4}{x^2-2x+1}=\frac{1}{4}$, to find x. *Ans.* $x=5$

28. Given $\dfrac{3x-1}{\sqrt{3x+1}}=1+\dfrac{\sqrt{3x}-1}{2}$ to find x. *Ans.* $x=3$

29. Given $\sqrt{x}+\sqrt{x-9}=\dfrac{36}{\sqrt{x-9}}$ to find x. *Ans.* $x=25$

30 Given $\dfrac{3\sqrt{x}-4}{\sqrt{x}+2}=\dfrac{3\sqrt{x}+15}{\sqrt{x}+40}$ to find x. *Ans.* $x=4$

31. Given $\dfrac{\sqrt{x}+\sqrt{x-a}}{\sqrt{x}-\sqrt{x-a}}=\dfrac{n^2a}{x-a}$ to find the value of x.

Ans. $x=a\dfrac{(n+1)^2}{2n+1}$.

Multiply the first member, numerator and denominator, by $(\sqrt{x}+\sqrt{x-a})$, then both members by a, and extract square root.

32. Given $\dfrac{a+x+\sqrt{2ax+x^2}}{a+x}=b$, to find x.

Assume $a+x=y$. Then the equation becomes

$$\frac{y+\sqrt{y^2-a^2}}{y}=b. \quad \text{Hence, } y=\frac{\pm a}{\sqrt{2b-b^2}}$$

$$\text{And} \qquad x=\pm a\left(\frac{1\pm\sqrt{2b-b^2}}{\sqrt{2b-b^2}}\right).$$

(Art. 90.) To resolve the following examples, requires a degree of tact not to be learned from rules. Quickness of *perception* is requisite, as well as sound reasoning. Quickness to perceive the form of *binomial* squares, and *binomial* cubes, and readiness to resolve quantities into simple or *compound* factors, as the case may require.

1. Given $x^2+2x=9+\frac{18}{x}$ to find the value of x

Multiply by x, and $x^3+2x^2=9x+18$.

Separate into factors, thus: $(x+2)x^2=(x+2)9$.

Divide by the common factor, $x+2$, and $x^2=9$, or $x=\pm3$.

2 Given $\left\{\begin{matrix} x^2+xy=12 \\ y^2+xy=24 \end{matrix}\right\}$ to find the values of x and y.

Add the two equations together, and we have

$$x^2+2xy+y^2=36.$$

Extract square root, and $x+y=\pm6$. (A)

From the first equation we have $(x+y)x=12$. (B)

Divide equation (B) by (A), and $x=\pm2$.

This example required perception to recognise the binomial square, and also to separate into factors.

3. Given $x^2+y^2=\frac{13}{x-y}$ and $xy=\frac{6}{x-y}$ to find the values of x and y.

From the first equation subtract twice the second, and

$$x^2-2xy+y^2=(x-y)^2=\frac{1}{x-y}$$

Therefore, $(x-y)^3=1$, and $x-y=1$.

Continuing the operation, we shall find $x=3$, and $y=2$.

4. Given $x^2y+xy^2=180$ and $x^3+y^3=189$, to find the values of x and y. *Ans.* $x=5$ or 4; $y=4$ or 5.

To resolve this problem, requires the formation of a cube, or to resolve quantities into factors.

5. Given $x^3+y^3=(x+y)xy$, and $x+y=4$, to find the values of x and y. *Ans.* $x=2$; $y=2$.

6. Given $x+y : x :: 7 \quad 5$, and $xy+y^2=126$, to find the values of x and y. *Ans.* $x=\pm15$, $y=\pm6$

7. Given $x-y:y::4:5$, and $x^2+4y^2=181$, to find the values of x and y. *Ans.* $x=\pm9$, $y=\pm5$.

8. Given $\sqrt{x}+\sqrt{y}:\sqrt{x}-\sqrt{y}::4:1$, and $x-y=16$, to find the values of x and y. *Ans.* $x=25$, $y=9$.

9 Given $\frac{x^2}{9}+\frac{x}{3}+\frac{1}{4}=9$, to find x. *Ans* $x=7\frac{1}{2}$.

10. $\left.\begin{array}{ll}\text{Given} & x+y:x-y::3:1\\ \text{And} & x^3-y^3=56\end{array}\right\}$ to find x and y.

Ans. $x=4$ $y=2$

11. $\left.\begin{array}{ll}\text{Given} & x^2y+xy^2=30\\ \text{And} & \frac{1}{x}+\frac{1}{y}=\frac{5}{6}\end{array}\right\}$ to find x and y.

Ans $x=3$, $y=2$

Observe that $xy(x+y)=x^2y+xy^2$.

Clear the 2d equation of fractions, and $y+x$ or $x+y=\frac{5xy}{6}$.

Now assume $x+y=s$, and $xy=p$. Then the original equations become $sp=30$

And $6s=5p$

Equations which readily give s and p, and from them we determine x and y.

N. B. When two unknown quantities, as x and y, produce equations in the form of

$$x+y=s \qquad (1)$$

And $$xy=p \qquad (2)$$

such equation can be resolved in the following manner:

Square (1), and $x^2+2xy+y^2=s^2$

Subtract 4 times (2) $4xy \quad =4p$

Diff. is $x^2-2xy+y^2=s^2-4p$

By evolution $x-y=\sqrt{s^2-4p}$ (3)

Add equation (1) and (3), and we have,

$$2x=s+\sqrt{s^2-4p} \qquad (4)$$

Sub. (3) from (1), and $2y=s-\sqrt{s^2-4p}$ (5)

To verify equations (4) and (5), add them and divide by 2, and we have $x+y=s$. Multiply (4) by (5), and divide by 4, and we have $xy=p$.

(Art. 91.) No person can become very skilful in algebraic operations as long as he feels averse to substitution; for judicious substitution stands in the same relation to common algebra, as algebra stands to arithmetic. The last example is an illustration of this remark. To acquire the habit of substituting, may require some extra attention at first, but the power and advantage gained will a thousand fold repay for all additional exertion.

When two equations involve quantities in the form $(x+y)$ and xy, or $x-y$, and xy, and any product of these quantities, the equations are easily solved.

The following examples will illustrate:

12. Given $x+\sqrt{xy}+y=19$ } to find x and y.
And $x^2+xy+y^2=133$ }

Put $x+y=s$, and $\sqrt{xy}=p$ $\therefore xy = p^2$

Then $s+p=19$ (A)

And $s^2-p^2=133$ (B)

Divide (B) by (A), and we have $s-p=7$, &c.

Ans. $x=9$ or 4, $y=4$ or 9.

13. Given $x^4+2x^2y^2+y^4=1296-4xy(x^2+xy+y^2)$ and $x-y=4$, to find x and y.

Put $x^2+y^2=s$, and $xy=p$

Then the first equation becomes $s^2=1296-4p(s+p)$

Multiply and transpose, and $s^2+4sp+4p^2=1296$

Square root $s+2p=\pm 36$

But $s+2p=x^2+2xy+y^2=\pm 36$

Therefore $x+y=\pm 6$, or $\pm\sqrt{-36}$.

Rejecting *imaginary* quantities, we find $x=5$ or -1, and $y=1$ or -5.

14. Given $\dfrac{x^2-y^2}{x-y}=6$, and $x+y+xy=11$, to find the values of x and y.

Ans. $x=5$ or 1, $y=1$ or 5

15. Given $x^3+y^3=2xy(x+y)$, and $xy=16$, to find the values of x and y. *Ans.* $x=2\sqrt{5}+2$, $y=2\sqrt{5}-2$.

16. Given $x^3+y^3=a$, and $x^2y+xy^2=a$, to find the relative values of x and y. *Ans.* $x=y$

17. Given $x+y : x :: 5 : 3$, and $xy=6$, to find x and y. *Ans.* $x=\pm3$, $y=\pm2$.

18. Given $x+y : x :: 7 : 5$, and $xy+y^2=126$, to find x and y. *Ans.* $x=\pm15$, $y=\pm6$.

19. Given $x^2+y^2=a$, and $xy=b$, to find the values of x and y. *Ans.* $x=\pm\frac{1}{2}\sqrt{a+2b}+\frac{1}{2}\sqrt{a-2b}$.
$y=\pm\frac{1}{2}\sqrt{a+2b}-\frac{1}{2}\sqrt{a-2b}$.

(Art. A)[*] Equations in the form of $x^4-2ax^2+a^2=b$, require for their complete solution, the square root of an expression in the form of $a\pm\sqrt{b}$; for by extracting the square root of the equation, we have

$$x^2-a=\pm\sqrt{b}$$

Hence $$x=\sqrt{a\pm\sqrt{b}}$$

The right hand member of this equation is an expression well known among mathematicians as

A BINOMIAL SURD.

Expressions in this form may or may not be complete powers; and it is very advantageous to extract the root of such as are complete, for the roots will be smaller, and more simple quantities, in the form of $a'\pm\sqrt{b'}$, or of $\sqrt{a'}\pm\sqrt{b'}$.

Let us now investigate a method of extracting these roots; and, for the sake of simplicity, let us square $3+\sqrt{7}$.

By the rule of squaring a binomial, we have $9+6\sqrt{7}+7$,

Or, $16+6\sqrt{7}$;

Conversely, then, the square root of $16+6\sqrt{7}$, is $3+\sqrt{7}$.

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[*] That the same Articles may number the same in both the School and College Edition, we shall designate all additional Articles, in this volume, by A, B, &c.

But when a root consists of two parts, *its square consists of the sum of the squares of the two parts, and twice the product of the two parts.*

Now we readily perceive that 16 is the sum of the squares of the two parts expressing the root; and $6\sqrt{7}$, the part containing the radical, is twice the product of the two parts.

To find what this root must be, let x represent one part of the root, and y the other:

Then $x^2+y^2=16$, (1)

And $2xy=6\sqrt{7}$, (2)

Add equations (1) and (2), and extract square root, and we have

$$x+y=\sqrt{16+6\sqrt{7}} \qquad (3)$$

Subtract equation (2) from (1), and extract square root, and we have

$$x-y=\sqrt{16-6\sqrt{7}}. \qquad (4)$$

Multiply (3) and (4), and we have

$$x^2-y^2=\sqrt{256-252}=\sqrt{4}=2; \quad (5)$$

Add (1) and (5), and we have

$$2x^2=18, \quad \text{or} \quad x=3;$$

Sub. (5) from (1), and $2y^2=14$, or $y=\sqrt{7}$.

Whence $x+y$, or the square root of $16+6\sqrt{7}$ is $3+\sqrt{7}$.

We shall now be more general.

Take two roots, one in the form of $a\pm\sqrt{b}$,

and one in the form of $\sqrt{a}\pm\sqrt{b}$;

Square both, and we shall have $a^2\pm2a\sqrt{b}+b$,

And $a\pm2\sqrt{ab}+b$.

In numerals, and, in short, in all cases, the sum of the squares of the two parts of the root, as (a^2+b), in the first square, and $(a+b)$, in the second, contains no radical sign; and the sum of these rational parts may be represented by c and c', and the squares represented in the form of

$$c \pm 2a\sqrt{b},$$

or of $$c' \pm 2\sqrt{ab}.$$

Hence, generally, if we represent the parts of the roots by x and y, we shall have $x^2+y^2=$ the sum of the rational parts, and $2xy=$ the term containing the radical.

The signs to x and y must correspond to the sign between the terms in the power. If that sign is minus, one of the signs of the root will be minus; it is indifferent which one.

EXAMPLES.

1. What is the square root of $11+6\sqrt{2}$? *Ans.* $3+\sqrt{2}$.

2. What is the square root of $7+4\sqrt{3}$? *Ans.* $2+\sqrt{3}$.

3. What is the square root of $7-2\sqrt{10}$?
Ans. $\sqrt{5}-\sqrt{2}$ or $\sqrt{2}-\sqrt{5}$.

4. What is the square root of $94+42\sqrt{5}$? *Ans.* $7+3\sqrt{5}$.

5. What is the square root of $28+10\sqrt{3}$? *Ans.* $5+\sqrt{3}$.

6. What is the square root of

$$np+2m^2-2m\sqrt{np+m^2}\,?$$

In this example put $a=np+m^2$, and x and y to represent the two parts of the root,

Then $$x^2+y^2=m^2+a,$$

and $$2xy=-2m\sqrt{a}.$$

Ans. $\pm\left(\sqrt{np+m^2}-m\right)$

7. What is the square root of $bc+2b\sqrt{bc-b^2}$?
Ans. $\pm\left(b+\sqrt{bc-b^2}\right)$

8. What is the sum of $\sqrt{16+30\sqrt{-1}}+\sqrt{16-30\sqrt{-1}}$?
Ans. 10.

9. What is the sum of $\sqrt{11+6\sqrt{2}}$ and $\sqrt{7-2\sqrt{10}}$?
Ans. $3+\sqrt{5}$.

10 What is the sum of $\sqrt{31+12\sqrt{-5}}$ and $\sqrt{-1+4\sqrt{-5}}$?
Ans. $8+2\sqrt{-5}$ or 4.

In a similar manner we may extract the cube root of a binomial surd, when the expression is a cube; but the general solution involves the solution of a cubic equation, and, of course, must be omitted at this place; and, being of little practical utility, we may omit it altogether.

(Art. 92.) Fractional exponents are at first very troublesome to young algebraists; but such exponents can always be banished from *pure* equations by *substitution.* For the exponents of all such equations must be multiples of each other; otherwise they would not be *pure*, but complex equations.

To make the proper substitution, place the unknown quantities having the lowest exponents equal to other simple quantities, as P and Q; and let this be a general direction.

EXAMPLES.

1. Given $x^{\frac{2}{3}}+y^{\frac{1}{5}}=6$, and $x^{\frac{4}{3}}+y^{\frac{2}{5}}=20$, to find the values of x and y

By the above direction, put $x^{\frac{2}{3}}=P$, and $y^{\frac{1}{5}}=Q$.

Squaring these auxiliaries, or assumed equations,

And $x^{\frac{4}{3}}=P^2$, and $y^{\frac{2}{5}}=Q^2$.

Now the original equations become

$$P+Q=6 \qquad (1)$$

$$P^2+Q^2=20 \qquad (2)$$

By squaring equation (1), $P^2+2PQ+Q^2=36$.

Subtracting equation (2), we have $2PQ=16$.

Subtracting this last from equation (2), and we have

$$P^2-2PQ+Q^2=4.$$

By extracting square root $P-Q=\pm 2$

But by equation (1), $\underline{P+Q=\ 6}$

Therefore, $P=4$ or 2, and $Q=2$ or 4,

Hence, $x^{\frac{2}{3}}=4$ or 2, and $y^{\frac{1}{5}}=2$ or 4.

$$\left.\begin{array}{l}\text{Square root } x^{\frac{1}{3}}=2 \text{ or } (2)^{\frac{1}{2}} \\ \text{Cubing gives } x=8 \text{ or } (2)^{\frac{3}{2}}\end{array}\right\} y=32 \text{ or } 1024$$

2. Given $xy^2+y=21$, and $x^2y^4+y^2=333$, to find the values of x and y.

By comparing exponents in the two equations, we perceive that if we put $xy^2=P$, and $y=Q$, the equations become

$$P+Q=21$$
$$P^2+Q^2=333$$

Solved as the preceding, gives $P=18$, $Q=3$.

From which we obtain $x=2$, or $\frac{1}{108}$, $y=3$ or 18.

3. Given $\left.\begin{array}{l} x^2+x^{\frac{4}{3}}y^{\frac{2}{3}}=208 \\ \text{And } y^2+x^{\frac{2}{3}}y^{\frac{4}{3}}=1053 \end{array}\right\}$ to find the values of x and y.

Assume $x^{\frac{2}{3}}=P$, and $y^{\frac{2}{3}}=Q$.

By squaring and cubing these assumed auxiliary equations, we have

$$x^{\frac{4}{3}}=P^2, \qquad y^{\frac{4}{3}}=Q^2,$$
$$x^2=P^3, \qquad y^2=Q^3.$$

Seek the common measure (if there be one) between 208 and 1053.

From the above substitution, the given equations become

$$P^3+P^2Q=208=13.16 \qquad (1)$$
$$Q^3+Q^2P=1053=13.81 \qquad (2)$$

Separate the left hand members into factors, and

$$P^2(P+Q)=13.16 \qquad (3)$$
$$Q^2(Q+P)=13.81 \qquad (4)$$

Divide equation (4) by (3), and we have

$$\frac{Q^2}{P^2}=\frac{81}{16}. \quad \text{Extracting square root} \quad \frac{Q}{P}=\frac{9}{4}$$

Or $Q=\frac{9P}{4}$. Substitute this value of Q in equation (1),

And $P^3+\frac{9P^3}{4}=13.16$

Or $4P^3+9P^3=13.64$

That is, $13P^3=13.64$

Hence, $P^3=64$ or $P=4$

But $x^2=P^3=64$. Therefore, $x=\pm 8$

As $Q=\frac{9P}{4}$ and $P=4$, $Q=9$ and $y=\pm 27$.

4. $$\left.\begin{array}{ll}\text{Given} & x^{\frac{3}{2}}+x^{\frac{3}{4}}y^{\frac{3}{4}}+y^{\frac{3}{2}}=1009\ \ =a\\ \text{And} & x^3+x^{\frac{3}{2}}y^{\frac{3}{2}}+y^3=582193=b\end{array}\right\}\text{ to find } x \text{ and } y.$$

$$\begin{array}{llll}\text{Put} & x^{\frac{3}{4}}=P & \text{and} & y^{\frac{3}{4}}=Q\\ \text{Then} & x^{\frac{3}{2}}=P^2 & \text{and} & y^{\frac{3}{2}}=Q^2\\ \text{And} & x^3=P^4 & \text{and} & y^3=Q^4\end{array}$$

Our equations then become

$$\left.\begin{array}{l}P^2+PQ+Q^2=a\\ P^4+P^2Q^2+Q^4=b\end{array}\right\}$$ Equations having no fractional exponents, and are of the same form as in Problem 12. (Art. 91.)

Ans. $x=81$ or 16, $y=16$ or 81.

5. $$\left.\begin{array}{ll}\text{Given} & x+x^{\frac{1}{2}}y^{\frac{1}{2}}=12\\ \text{And} & y+x^{\frac{1}{2}}y^{\frac{1}{2}}=\ 4\end{array}\right\}\text{ to find the values of } x \text{ and } y.$$

Ans. $x=9$, $y=1$

6. $$\left.\begin{array}{ll}\text{Given} & x+x^{\frac{1}{2}}y^{\frac{1}{2}}=a\\ \text{And} & y+x^{\frac{1}{2}}y^{\frac{1}{2}}=b\end{array}\right\}\text{ to find the values of } x \text{ and } y.$$

Ans. $x=\frac{a^2}{a+b}$ $y=\frac{b^2}{a+b}$

7. $$\left.\begin{array}{ll}\text{Given} & x^{\frac{3}{2}}+x^{\frac{3}{4}}y^{\frac{3}{4}}=a\\ \text{And} & y^{\frac{3}{2}}+x^{\frac{3}{4}}y^{\frac{3}{4}}=b\end{array}\right\}\text{ to find the values of } x \text{ and } y$$

Ans. $x=\left(\frac{a^4}{(a+b)^2}\right)^{\frac{1}{3}}$ $y=\left(\frac{b^4}{(a+b)^2}\right)^{\frac{1}{3}}$.

8. Given $\sqrt{x}+\sqrt{y} : \sqrt{x}-\sqrt{y} :: 4 : 1$, and $x-y=16$, to find the values of x and y. *Ans.* $x=25$, $y=9$.

9 $$\left.\begin{array}{ll}\text{Given} & x^{\frac{1}{2}}+y^{\frac{1}{2}}=\ 5\\ \text{And} & x\ +y\ =13\end{array}\right\}\text{ to find the values of } x \text{ and } y.$$

Ans. $x=9$ or 4, $y=4$ or 9

10. Given $x+y : x-y :: 3 : 1$ And $x^2-y^2=56$ } to find the values of x and y.

Ans. $x=4$, $y=2$.

11. Given $x+y=35$ And $x^{\frac{1}{3}}+y^{\frac{1}{3}}=5$ } to find the values of x and y.

Ans. $x=27$, $y=8$.

CHAPTER V.

Problems producing Pure Equations.

(Art. 93.) We again caution the pupil, to be very careful not to involve factors, but keep them separate as long as possible, for greater simplicity and brevity. The solution of one or two of the following problems will illustrate.

1. It is required to divide the number of 14 into two such parts, that the quotient of the greater divided by the less, may be to the quotient of the less divided by the greater, as 16 : 9.

Ans. The parts are 8 and 6.

Let $x=$ the greater part. Then $14-x=$ the less.

Per question, $\frac{x}{14-x} : \frac{14-x}{x} :: 16 : 9.$

Multiply extremes and means, and $\frac{9x}{14-x}=\frac{16(14-x)}{x}$

Clearing of fractions, we have $9x^2=16(14-x)^2$

By evolution, $3x=4(14-x)=4.14-4x$

By transposition, $7x=4.14$

By division, $x=4.2=8$, the greater part.

Had we actually multiplied $\frac{14-x}{x}$ by 16, in place of indicating it, the exact value and form of the factors would have been lost to view, and the solution might have run into an *adfected* quadratic equation.

The same remark may be applied to many other problems, and many are put under the head of quadratics that may be reduced by pure equations.

2. Find two numbers, whose difference, multiplied by the difference of their squares, is 32, and whose sum, multiplied by the sum of their squares, gives 272.

If we put $x=$ the greater, and $y=$ the less, we shall have

$$(x-y)(x^2-y^2)=32 \qquad (1)$$

And $$(x+y)(x^2+y^2)=272 \qquad (2)$$

Multiply these factors together, as indicated, and add the equations together, and divide by 2, and we shall have

$$x^3+y^3=152 \qquad (3)$$

If we take (1) from (2), after the factors are multiplied, we shall have $2xy^2+2x^2y=240$, or $xy(x+y)=120$ (4)

Three times equation (4) added to equation (3) will give a cube, &c. A better solution is as follows:

Let $x+y=$ the greater number, and $x-y=$ the less.

Then $2x=$ their sum, and $2y=$ their difference.

Also, $4xy$ is equal the difference of their squares, and $2x^2+2y^2=$ the sum of their squares.

By the conditions, $$2y\times 4xy=32$$

And $$2x(2x^2+2y^2)=272$$

By reduction, $$xy^2=4$$

And $$x^3+xy^2=68$$

By subtraction, $$x^3=64 \quad \text{or} \quad x=4$$

Hence, $y=1$, and the numbers are 5 and 3.

We give these two methods of solution to show how much depends on skill in taking first assumptions.

3. From two towns, 396 miles asunder, two persons, *A* and *B*, set out at the same time, and met each other, after traveling as many days as are equal to the difference of miles they traveled per day, when it appeared that *A* had traveled 216 miles. How many miles did each travel per day? Let $x=A$'s rate, and $y=B$'s rate.

Then $x-y=$ the days they traveled before meeting.

By question, $(x-y)x=216$, and $(x-y)y=180$.

Consequently, $\frac{216}{x}=\frac{180}{y}$ or $\frac{6}{x}=\frac{5}{y}$.

Therefore, $y=\frac{5}{6}x$, which substitute in the first equation, and we have $(x-\frac{5}{6}x)x=216$, or $\frac{x^2}{6}=216=6\times 6\times 6$.

By evolution, $x=36$; therefore $y=30$.

4. Two travelers, *A* and *B*, set out to meet each other, *A* leaving the town *C*, at the same time that *B* left *D*. They traveled the direct road between *C* and *D*; and on meeting, it appeared that *A* had traveled 18 miles more than *B*, and that *A* could have gone *B*'s distance in 15¾ days, but *B* would have been 28 days in going *A*'s distance. Required the distance between *C* and *D*.

Let $x=$ the number of miles *A* traveled.

Then $x-18=$ the number *B* traveled.

$$\frac{x-18}{15\frac{3}{4}}=A\text{'s daily progress.}$$

$$\frac{x}{28}=B\text{'s daily progress.}$$

Therefore, $$x : x-18 :: \frac{x-18}{15\frac{3}{4}} : \frac{x}{28}$$

And $$\frac{x^2}{28}=\frac{4(x-18)^2}{63}.$$

Divide the denominators by 7, and extract square root, and we have

$$\frac{x}{2}=\frac{2}{3}(x-18).$$

Therefore, $x=72$; and the distance between the two towns is 126 miles.

5. The difference of two numbers is 4, and their sum multiplied by the difference of their second powers, gives 1600. What are the numbers? *Ans.* 12 and 8.

6. What two numbers are those whose difference is to the less as 4 to 3, and their product, multiplied by the less, is equal to 504? *Ans.* 14 and 6.

7. A man purchased a field, whose length was to its breadth as 8 to 5. The number of dollars paid per acre was equal to the number of rods in the length of the field; and the number of dollars given for the whole was equal to 13 times the number of rods round the field. Required the length and breadth of the field.

Ans. Length 104 rods, breadth 65 rods.

Put $8x$=the length of the field.

8. There is a stack of hay, whose length is to its breadth as 5 to 4, and whose height is to its breadth as 7 to 8. It is worth as many cents per cubic foot as it is feet in breadth; and the whole is worth at that rate 224 times as many cents as there are square feet on the bottom. Required the dimensions of the stack.

Put $5x$ = the length.

Ans. Length 20 feet, breadth 16 feet; height 14 feet.

9. There is a number, to which if you add 7, and extract the square root of the sum, and to which if you add 16 and extract the square root of the sum, the sum of the two roots will be 9. What is the number? *Ans.* 9.

Put $x^2-7=$ the number.

10. *A* and *B* carried 100 eggs between them to market, and each received the same sum. If *A* had carried as many as *B*, he would have received 18 pence for them; and if *B* had taken as many as *A*, he would have received 8 pence. How many had each? *Ans.* *A* 40, and *B* 60.

11. The sum of two numbers is 6, and the sum of their cubes is 72. What are the numbers? *Ans.* 4 and 2.

12. One number is a^2 times as much as another, and the product of the two is b^2. What are the numbers?

Ans. $\frac{b}{a}$ and ab.

13. The sum of two numbers is 100, the difference of their square roots is 2. What are the numbers?

Ans. 36 and 64.

Put $x=$ the square root of the greater number,
And $y=$ the square root of the less number; or
Put $x+y=$ the square root of the greater, &c.

14. It is required to divide the number 18 into two such parts, that the squares of those parts may be to each other as 25 to 16

Let $x=$ the greater part. Then $18-x=$ the less.

By the condition proposed, $x^2:(18-x)^2::25:16$.

Therefore, $16x^2=25(18-x)^2$

By evolution, $4x=\pm 5(18-x)$

If we take the plus sign, as we must do by the strict enunciation of the problem, we find $x=10$. Then $18-x=8$.

And $(10)^2:(8)^2::25:16$

If we take the minus sign, we shall find $x=90$.

Then $18-x=18-90=-72$.

And $(90)^2:(-72)^2::25:16$; a true proportion, corresponding to the enunciation; but 18 in this case is not the number divided, it is the *difference* between two numbers whose squares are in proportion of 25 to 16.

15. It is required to divide the number a into two such parts that the *squares* of those parts may be in proportion of b to c.

Let $x=$ one part, then $a-x=$ the other.

By the condition, $x^2:(a-x)^2::b:c$

Therefore, $cx^2=b(a-x)^2$

By evolution, $\sqrt{c}x=\pm\sqrt{b}(a-x)$

Taking the plus sign, $x=\dfrac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}$ and $a-x=\dfrac{a\sqrt{c}}{\sqrt{b}+\sqrt{c}}$

Taking the minus sign, $x=\dfrac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}$ and $a-x=\dfrac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}}$.

Prob. 14, is a particular case of this general problem, in which $a=18, b=25$, and $c=16$; and substituting these values in the result, we find $x=10$, and $x=90$, as before.

If we take $b=c$, the two divisions will be equal, each equal to $\frac{1}{2}a$, when the plus sign is used; but when the minus sign is used, $x=\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}=\frac{a\sqrt{b}}{0}$, a symbol of infinity, as the denominator is contained in the numerator an infinite number of times. (Art. 58.) The other part, $a-x=\frac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}}=\frac{-a\sqrt{c}}{0}$, also a symbol of infinity; and the two parts,

$$\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}-\frac{a\sqrt{c}}{\sqrt{b}-\sqrt{c}}=\frac{a(\sqrt{b}-\sqrt{c})}{(\sqrt{b}-\sqrt{c})}=a.$$

It may appear absurd, that the two parts, both infinite and having a ratio of equality, (which they must have, if $b=c$) can still have a difference of a. But this apparent absurdity will vanish, when we consider that the two parts being infinite in comparison to our standards of measure, can have a difference of any finite quantity which may be great, compared with small standards of measure, but becomes nothing in comparison with infinite quantities. See (Art. 60.)

Application of the foregoing Problem.

(Art. 94.) It is a well established principle in physics that *light* and *gravity* emanating from any body, diminish in *intensity* as the square of the distance increases.

Two bodies at a distance from each other, and attracting at a given point, their intensities of attraction will be to each other as the masses of the bodies directly and the squares of their distances inversely. Two lights, at a distance from each other, illuminating at a given point, will illuminate in proportion to the magnitudes of the lights directly, and the squares of their distances inversely.

These principles being admitted—

16. Whereabouts on the line between the earth and the moon will these two bodies attract equally, admitting the mass of the earth to be 75 times that of the moon, and their distance asunder 30 diameters of the earth?

Represent the mass of the moon by c,
and the mass of the earth by b,
their distance asunder by a.

The distance of the required point from the earth's centre, represent by x. Then the remaining distance will be $(a-x)$.

Now by the principle above cited, $x^2 : (a-x)^2 :: b : c$.

This proportion is the same as appears in the preceding general problem: except that we have here actually made the application, and must give the definite values to a, b and c.

As before, $x=\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}$ and $a-x=\frac{a\sqrt{c}}{\sqrt{b}+\sqrt{c}}$

$$a=30, \qquad b=75, \qquad c=1.$$

$x=\frac{30\sqrt{75}}{\sqrt{75}+1}=26.9$, nearly. Hence, $a-x=3.1$, nearly.

If we take the second values for the two distances, from the general result, namely, $x=\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}$ and $a-x=\frac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}}$, and give the numeral values, we shall have

$$x=\frac{30\sqrt{75}}{\sqrt{75}-1}=33.9, \quad a-x=-3.9, \text{ nearly.}$$

These values show that in a line beyond the moon, at a distance of 3.9 the diameters of the earth, a body would be attracted as much by the earth as by the moon, and the value of $(a-x)$ being minus, shows that the distance is now counted the other way from the moon, not as in the first case towards the earth; and the real distance, 30, corresponding to a in the general problem, is now a *difference*.

We may make very many inquiries concerning the intensity of attraction on this line, on the same general principle.

For example, *we may inquire, whereabouts, on the line between the earth and moon will the attraction of the earth be* 16 *times the attraction of the moon?*

Let $x=$ the distance from the earth.

Then $a-x=$ the distance from the moon.

The attraction of the earth will be represented by $\frac{b}{x^2}$.

The attraction of the moon at the same point will be $\frac{c}{(a-x)^2}$

By the question. $\frac{b}{x^2}=\frac{16c}{(a-x)^2}$

By evolution, $\frac{\sqrt{b}}{x}=\pm\frac{4\sqrt{c}}{a-x}$

Clearing of fractions, $a\sqrt{b}-\sqrt{b}x=4\sqrt{c}x.$

Using the plus sign, $x=\frac{a\sqrt{b}}{\sqrt{b}+4\sqrt{c}}=20.5$, nearly.

Using the minus sign, $x=\frac{a\sqrt{b}}{\sqrt{b}-4\sqrt{c}}=55.7$, nearly, or 25.7 diameters of the earth beyond the moon.

Observe that the 4 which stands as a factor to $\sqrt{c}$ is the square root of 16, the number of times the intensity of the earth's attraction was to exceed that of the moon.

If we propose any other number in the place of 16, its square root will appear as a factor to $\sqrt{c}$; we may therefore inquire at what distance the intensity of the earth's attraction will be n times that of the moon, and the answer will be from the earth in a line through the moon,

$$\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{nc}} \quad \text{and} \quad \frac{a\sqrt{b}}{\sqrt{b}-\sqrt{nc}}.$$

The same application that we have made of this general problem to the two bodies, the earth and the moon, may be made to any two bodies in the solar system; and the same application we have made to attraction may be made to light, whenever we can decide the relative intensity of any two lights at any assumed *unity* of distance.

(Art. 95.) This problem may be varied in its application to meet cases where the distances are given, and the *comparative* intensities of light or attraction are required.

For example, the planet *Mars* and the *moon* both transmit the sun's light to the earth by reflection, and we now inquire the relative intensities of their lights at *given distances*, and in given positions.

If the surface of Mars and that of the moon were equal, they would receive the same light from the sun at equal distance from that luminary; but at different distances equal surfaces would receive light reciprocally proportional to the squares of their distances.

The surfaces of globular bodies are in proportion to the squares of their diameters. Now let M represent the diameter of Mars and m the diameter of the moon. Also, let R represent the distance of Mars from the sun, and r the distance of the moon from the sun.

Then the quantity of light received by Mars may be expressed by $\frac{M^2}{R^2}$; and the relative quantity received by the moon by $\frac{m^2}{r^2}$. But these lights, when reflected to the earth, must be diminished by the squares of the distances of these two bodies from the earth. Now if we put D to represent the distance of Mars from the earth, and d the distance of the moon, we shall have $\frac{M^2}{R^2D^2}$ for the relative illumination by Mars when the whole enlightened face of that planet is towards the earth, and $\frac{m^2}{r^2d^2}$ for the light of the full moon.

When the whole illuminated side of Mars is turned towards the earth, which is the case under consideration, (if we take the whole diameter of the body,) it is then in opposition to the sun, and gives us light, we know not how much, as we have no standard of measure for it; but we can make a comparative measure of one by the other, and therefore the light of Mars in this position may be taken as *unity*, and in comparison with this let us call the light of the full moon x.

$$\text{Then } \frac{M^2}{R^2D^2} : \frac{m^2}{r^2d^2} :: 1 : x$$

$$\text{Therefore } x=\left(\frac{m^2}{M^2}\right)\left(\frac{R^2}{r^2}\right)\left(\frac{D^2}{d^2}\right).$$

As the value of a fraction depends only on the relation of the numerator to the denominator, to find the numeral value of x, it will be sufficient to seek the relation of m to M, of R to r, and of D to d.

M=4000 miles nearly, and m=2150; hence, $\frac{m}{M}=\frac{43}{80}$

R=144000000, and r=95000000; or $\frac{R}{r}=\frac{144}{95}$

D=144000000—95000000=49000000 or $\frac{D}{d}=\frac{4900}{24}$

d=240000

$$\text{Therefore,}\quad x=\left(\frac{43}{80}\right)^2\left(\frac{144}{95}\right)^2\left(\frac{4900}{24}\right)^2=27611$$

That is, in round numbers, the light of the full moon is twenty-seven thousand six hundred times the light of Mars, when that planet is brightest, in its opposition to the sun.

We will add one more example by the way of farther illustration.

What comparative amount of solar light is reflected to the earth by Jupiter and Saturn, when those planets are in opposition to the sun;—the relative diameter of Jupiter being to that of Saturn as 111 to 83, and the *relative* distances of the Earth, Jupiter and Saturn, from the sun, being as 10, 52 and 95, respectively?

Ans. Taking the light reflected by Saturn for unity, that by Jupiter will be expressed by $24.\frac{45}{100}$ nearly.

The philosophical student will readily perceive a more extended application of these principles to computing the *relative* light reflected to us by the different planets; but we have gone to the utmost limit of propriety, in an elementary work like this.

From Art. 94th to the end of this chapter can hardly be said to be algebra; it is natural philosophy, in which the science of algebra is used; however, we would offer no apology for thus giving a glimpse of the utility, the *cui bono*, and the application of algebraic science.

SECTION IV.

QUADRATIC EQUATIONS.

CHAPTER I.

(Art. 96.) Quadratic equations are either simple or compound. A simple quadratic is that which involves the square of the unknown quantity only, as $ax^2=b$; which is one form of pure equations, such as have been exhibited in the preceding chapter.

Compound quadratics, or, as most authors designate them, *adfected quadratics*, contain both the square and the first power of the unknown quantity, and of course cannot be resolved as simple equations.

All compound quadratic equations, when properly reduced, may fall under one of the four following forms :

$$(1)\ x^2+2ax=b$$
$$(2)\ x^2-2ax=b$$
$$(3)\ x^2-2ax=-b$$
$$(4)\ x^2+2ax=-b$$

If we take $x+a$ and square the sum, we shall have

$$x^2+2ax+a^2$$

If we take $x-a$ and square, we shall have

$$x^2-2ax+a^2$$

If we reject the 3rd terms of these squares, we have

$$x^2+2ax, \text{ and } x^2-2ax$$

The same expressions that we find in the first members of the four preceding theoretical equations.

It is therefore obvious that by adding a^2 to both sides of the preceding equations, the first members become *complete* squares. But in numeral quantities how shall we find the quantity corresponding to a^2? We may obtain a^2 by the formal process of taking half the coefficient of the first power of x, or the half of $2a$ or $-2a$, which is a or $-a$, the square of either being a^2.

Hence, when any equation appears in the form of $x^2\pm 2ax=\pm b$, we may render the first member a complete square, and effect a solution by the following

RULE. *Add the square of half the coefficient of the lowest power of the unknown quantity to the first member to complete its square; add the same to the second member to preserve the equality.*

Then extract the square root of both members, and we shall have equations in the form of

$$x\pm a=\pm\sqrt{b+a^2}$$

Transposing the known quantity a and the solution is accomplished.

In this manner we find the values of x in the four preceding equations, as follows:

$$(1)\quad x=-a\pm\sqrt{b+a^2}$$
$$(2)\quad x=\ \ a\pm\sqrt{b+a^2}$$
$$(3)\quad x=\ \ a\pm\sqrt{a^2-b}$$
$$(4)\quad x=-a\pm\sqrt{a^2-b}$$

When b is greater than a^2 equations (3) and (4) require the square root of a negative quantity, and there being no roots to negative quantities, the values of x in such cases are said to be *imaginary*.

The double sign is given to the root, as both plus and minus will give the same power, and this gives rise to two values of the unknown quantity; either of which substituted in the original equation will verify it.

After we reduce an equation to one of the preceding forms, the solution is only substituting particular values for a and b; but in many cases it is more easy to resolve the equation as an original one, than to refer and substitute from the formula.

(Art. 97.) We may meet with many quadratic equations that would be very inconvenient to reduce to the form of $x^2+2ax=b$; for when reduced to that form $2a$ and b may both be troublesome fractions.

Suck equations may be left in the form of

$$ax^2+bx=c$$

An equation in which the known quantities, a, b, and c, are all *whole numbers*, and at least a and b prime to each other.

We now desire to find some method of making the first member of this equation a square, without making fractions. We therefore cannot divide by a, because b is not divisible by a, the two letters being prime to each other by hypothesis. But the first term of a binomial square is always a square. Therefore, if we desire the first member of our equation to be converted into a binomial square, we must render the first term a square, and we can accomplish that by multiplying every term by a.

The equation then becomes

$$a^2x^2+bax=ca$$

Put $y=ax$. Then $y^2+by=ca$

Complete the square by the preceding rule, and we have

$$y^2+by+\frac{b^2}{4}=ca+\frac{b^2}{4}.$$

We are sure the first member is a square; but one of the terms is fractional, a condition we wished to avoid; but the denominator of the fraction is 4, a *square*, and a square multiplied by a square produces a square.

Therefore, multiply by 4, and we have the equation

$$4y^2+4by+b^2=4ca+b^2$$

An equation in which the first member is a binomial square and not fractional.

If we return the values of y and y^2 this last equation becomes

$$4a^2x^2+4abx+b^2=4ac+b^2$$

Compare this with the primitive equation

$$ax^2+bx=c.$$

We multiplied this equation first by a, then by 4, and in addition to this we find b^2 on both sides of the *rectified* equation, b being the coefficient of the first power of the unknown quantity. From this it is obvious that to convert the expression ax^2+bx into a binomial square, we may use the following

RULE 2. *Multiply by four times the coefficient of* x^2, *and add the square of the coefficient of* x.

To preserve equality, both sides of an equation must be multiplied by the same factors, and the same additions to both sides. We operate on the first member of an adfected equation to make it a *square*, we operate on the second member to preserve *equality*.

(Art. 98.) For the following method of avoiding fractions in completing the square, the author is indebted to the late Professor T. J. Matthews, of Ohio.

Resume the general equation $ax^2+bx=c$

Assume $x=\frac{u}{a}$ Then $ax^2=\frac{u^2}{a}$ and $bx=\frac{bu}{a}$

The general equation becomes $\frac{u^2}{a}+\frac{bu}{a}=c$

Or $u^2+bu=ac$

Now when b is even, we can complete the square by the first rule without making a fraction. In such cases this transformation is very advantageous.

When b is not even, multiply the general equation by 2, and the coefficient of x becomes even, and we have

$$2ax^2+2bx=2c \qquad (1)$$

Assume $x=\frac{u}{2a}$ Then $2ax^2=\frac{u^2}{2a}$ and $2bx=\frac{2bu}{2a}$

With these terms, equation (1) becomes

$$\frac{u^2}{2a}+\frac{2bu}{2a}=2c$$

Or $u^2+2bu=4ac$

Complete the square by the first rule, and we have

$$u^2+2bu+b^2=4ac+b^2$$

An equation essentially the same as that obtained by completing the square by the rule under (Art. 97.); for we perceive the second member is the same as would result from that rule; hence this method has no superior advantage except when b is even, in the first instance.

(Art. 99.) The foregoing rules are all that are usually given for the resolution of quadratic equations; ***but there are some*** *intricate cases in practice that we may meet with, where neither of the preceding rules appear practical or convenient.* To master these with skill and dexterity, we must return to a more general and comprehensive knowledge of *binomial squares.*

$x^2+2ax+a^2$ is a simple and complete binomial square. Let us strictly examine it, and we shall perceive,

1st. That it consists of *three* terms;

2d. Two of its terms, the *first* and the *third*, are *squares;*

3d. The middle term is twice the product of the square roots of the first and last term.

Now let us suppose the third term, a^2, to be lost, and we have only x^2+2ax. We know these two terms cannot make a square, as a binomial square must consist of three terms.*

We know also that the last term must be a square.

Let it be represented by t^2.

Then, by hypothesis, $x^2+2ax+t^2$ is a complete binomial square.

It being so, $2xt=2ax$, by the third observation above.

Therefore, $t=a$ and $t^2=a^2$

Thus a^2 is brought back.

1. Again, $4a^2+4ab$ are the first and second terms of a binomial square; what is the 3rd term?

Let t^2 represent the third term.

Then $4a^2+4ab+t^2$ is a binomial square.

Hence, $4at=4ab$ or $t=b$ and $t^2=b^2$

That is, t^2 represented the 3d term, and b^2 is the identical 3d term, and $4a^2+4ab+b^2$ is the actual binomial square whose root is $2a+b$.

2. $36y^2+36y$ are the first and 2d terms of a binomial square, what is the 3d term? *Ans.* 9.

3. $+\frac{6}{x}+9$ are the 2d and 3d terms of a square, what is the first? *Ans.* $\frac{1}{x^2}$.

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* In binomial surds two terms may make a square, and this may condemn the technicality here assumed; but it is nothing against the spirit of this article.

4. $\frac{49x^2}{4}-49$ are the 1st and 2d terms of a binomial square, what is the 3d? *Ans.* $\frac{49}{x^2}$.

5. $9y^2-6y$ are the 1st and 2d terms of a binomial square, what is the 3d? *Ans.* 1.

6. a^2x^2+bx are the 1st and 2d terms of a binomial square, what is the 3d? *Ans.* $\frac{b^2}{4a^2}$

7. $81x^2 \ \frac{1}{x^2}$ are the 1st and 3d terms of a binomial square, what is the 2d or middle term? *Ans.* ± 18.

8. $y^2-8x^{\frac{1}{2}}y$ are the 1st and 2d terms of a binomial square, what is the 3d? *Ans.* $16x$

9. $-\frac{12x}{19}+36$ are the 2d and 3d terms of a binomial square, what is the 1st? *Ans.* $\frac{x^2}{361}$.

10. $\frac{y^2}{361}+36$ are the 1st and 3d terms of a binomial square, what is the middle term? *Ans.* $\pm\frac{12y}{19}$

11. If $x+\frac{1}{16}$ are the 2d and 3d terms of a binomial square, what is the 1st term? *Ans.* $4x^2$.

12. The 1st term of a binomial square is $\frac{9x^2}{y^2}$ the 2d term is ± 12, what is the 3d term? *Ans.* $\frac{4y^2}{x^2}$.

(Art. 100.) Adfected quadratic equations, after being reduced to the form of $x^2+2ax=b$, can be resolved without any formality of completing the square, by the following substitution:

Assume $x=y-a$

Then $x^2=y^2-2ay+a^2$

And $2ax=\quad+2ay-2a^2$

By addition, $x^2+2ax=y^2-a^2=b$

Hence, $y=\pm\sqrt{b+a^2}$

And $x=-a\pm\sqrt{b+a^2}$, the same result as may be found in equation (1), (Art. 96.)

RULE FOR SUBSTITUTION. *Assume the value of the unknown quantity equal to another unknown, annexed to half the coëfficient of the inferior power with a contrary sign.*

(Art. 101.) For further illustration of the nature of quadratic equations, we shall work and discuss the following equation:

Given $x^2+4x=60$, to find x.

Completing the square, (Rule 1st.) $x^2+4x+4=64$.

Extracting square root, $x+2=\pm8$.

Hence, $x=6$ or $x=-10$.

That is, either *plus* 6, or *minus* 10, substituted for x in the given equation, will verify it.

For $6^2+4\times6=60$. Also, $(-10)^2-4\times10=60$

If $x=6$ then $x-6=0$

If $x=-10$ then $x+10=0$

Multiply these equations together, and we have

$$\begin{array}{rl} & x-6 \\ & x+10 \\ \hline & x^2-6x \\ & \quad\;\; 10x-60 \\ \hline \text{Product,} & x^2+4x-60=0 \end{array}$$

Transpose, and $x^2+4x=60$, the original equation.

Thus we perceive, that a quadratic equation may be considered as the product of two simple equations, and these values of x in the simple equations are said to be *roots* of the quadratic, and this view of the subject gives the *rationale* of the unknown quantity having *two* values.

In equations where but one value can be found, we infer that the other value is the same, and the two roots equal, or one of them a cipher.

EXAMPLES FOR PRACTICE.

1. Given $x^2-6x-7=33$, to find x. *Ans.* $x=10$ or -4.

2. Given $x^2-20x=-96$, to find x. *Ans.* 12 or 8.

3. Given $x^2+6x+1=92$, to find x. *Ans.* 7 or -13.

4. Given $y^2+12y=589$, to find y. *Ans.* 19 or -31.

5. Given $y^2-6y+10=65$, to find y. *Ans.* 11 or -5.

6. Given $x^2+12x+2=110$, to find x. *Ans.* 6 or -18.

7. Given $x^2-14x=51$, to find x. *Ans.* 17 or -3.

8. Given $x^2+6x+6=9$, to find x. *Ans.* $-3\pm2\sqrt{3}$.

9. Given $x^2+8x=12$, to find x. *Ans.* $-4\pm2\sqrt{7}$.

10. Given $x^2+12x=10$, to find x. *Ans.* $-6\pm\sqrt{46}$.

The reader will observe that the preceding examples are in, or can be immediately reduced to the form of $x^2\pm2ax=b$, and of course their solution is comparatively easy. The following are mostly in the form of $ax^2+bx=c$.

11. Given $5x^2+4x=204$, to find x.

According to (Art. 98,) put $x=\frac{u}{5}$. Then $5x^2=\frac{u^2}{5}$ and $4x=\frac{4u}{5}$, and the equation becomes $\frac{u^2}{5}+\frac{4u}{5}=204$.

Clearing of fractions, $u^2+4u=1020$.

Completing the square and extracting the root, we have,

$$u+2=\pm32, \quad \text{or} \quad u=30 \quad \text{or} \quad -34$$

But $x=\frac{u}{5}$. Therefore, $x=6$ or $-\frac{34}{5}$, *Ans.*

12. Given $5x^2+4x=273$, to find x. *Ans.* 7 or $-7\frac{4}{5}$

13. Given $7x^2-20x=32$, to find x. *Ans.* 4 or $-\frac{8}{7}$.

14. Given $25x^2-20x=-3$, to find x. *Ans.* $\frac{3}{5}$ or $\frac{1}{5}$.

15. Given $21x^2-292x=-500$, to find x.

Ans. $11\frac{19}{21}$ or 2

16. Given $2x^2—5x=117$, to find x.

Here, as 5 or b of the general equation is not even, we must multiply the whole equation by 2, to apply the above principle; or we may take Rule 2. (Art. 97.)

Multiply by 8, and add 5^2 or 25 to both members

Then $\quad 16x^2—40x+25=961$

Square root, $4x—5=\pm 31$. Hence, $x=9$ or $—6\frac{1}{2}$.

(Art. 102.) It should be observed that all quadratic equations can be reduced to the form of $x^2\pm 2ax=b$, or, as most authors give it, $x^2\pm px=q$; but when the terms would become fractional by such reduction, we prefer the form $ax^2\pm bx=\pm c$, for the sake of practical convenience, as mentioned in (Art. 97.)

(Art. 103.) It is not essential that the unknown quantity should be involved literally to its first and second powers; it is only essential that one index should be double that of the other. In such cases the equations can be resolved as quadratics. For example, $x^6—4x^3=621$ is an impure equation of the sixth degree, yet with a view to its solution, it may be called a quadratic. For we can assume $y=x^3$; then $y^2=x^6$, and the equation becomes $y^2—4y=621$, a quadratic in relation to y, giving $y=27$, or $—23$.

Therefore, $\quad x^3=27$ or $—23$

And $\quad x=3$ or $\sqrt[3]{—23}$.

There are other values of x; but it would be improper to seek for them now; such inquiries belong to the higher order of equations.

For another example, take $x^3—x^{\frac{3}{2}}=56$, to find the values of x.

Here we perceive one exponent of x is *double* that of the other, it is therefore essentially a quadratic.

Such cases can be made clear by assuming the lowest power of the unknown quantity equal to any simple letter. In the present case assume $y=x^{\frac{3}{2}}$; then $y^2=x^3$, and the equation is

$$y^2-y=56$$

By Rule 2, $4y^2-4y+1=225$

By evolution, $2y-1=\pm 15$

Hence, $y=8$ or -7

And by returning to the assumption $y=x^{\frac{3}{2}}$ we find $x^{\frac{3}{2}}=8$, or $x^{\frac{1}{2}}=2$. Hence, $x=4$; or, by taking the minus value of y, $x=\sqrt[3]{49}$.

(Art. 104.) When a compound quantity appears under different powers or fractional exponents, one exponent being double that of the other, we may put the quantity equal to a single letter, and make its quadratic form apparent and simple. For example, suppose the values of x were required in the equation

$$2x^2+3x+9-5\sqrt{2x^2+3x+9}=6$$

Assume $\sqrt{2x^2+3x+9}=y$

Then by involution, $2x^2+3x+9=y^2$ (A)

And the equation becomes $y^2-5y=6$ (B)

Which equation gives $y=6$ or -1. These values of y, substituted for y in the equation (A) give $2x^2+3x+9=36$

Or $2x^2+3x+9=1$

From the first of these we find $x=3$ or $-4\frac{1}{2}$

From the last we find $x=\frac{1}{4}(-3\pm\sqrt{-55},)$ imaginary quantities.

EXAMPLES.

1. Given $(x+12)^{\frac{1}{2}}+(x+12)^{\frac{1}{4}}=6$ to find the values of x.

Ans. $x=4$ or 69

2. Given $(x+a)^{\frac{1}{2}}+2b(x+a)^{\frac{1}{4}}=3b^2$, to find the values of x

Ans. $x=b^4-a$ or $81b^4-a$.*

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* It is proper to remark, that in many instances it would be difficult to verify the equation by taking the second values of x, as by squaring, the minus quantity becomes plus, and in returning the values, there is no method but trial to decide whether we shall take a plus or a minus root. Hence, these second answers are sometimes called roots of solution. In many instances hereafter, we shall give the rational and positive root only.

3. Given $9x+4+2\sqrt{9x+4}=15$, to find the values of x.
Ans. $x=\frac{5}{9}$ $\frac{7}{3}$.

4. Given $(10+x)^{\frac{1}{2}}-(10+x)^{\frac{1}{4}}=2$, to find x
Ans. $x=6$.

5. Given $(x-5)^3-3(x-5)^{\frac{3}{2}}=40$, to find x.
Ans. $x=9$.

6. Given $2(1+x-x^2)-(1+x-x^2)^{\frac{1}{2}}+\frac{1}{9}=0$, to find x.
Ans. $x=\frac{1}{2}+\frac{1}{6}\sqrt{41}$.

7. Given $x+16-3(x+16)^{\frac{1}{2}}=10$, to find x. *Ans.* $x=9$

8. Given $3x^{2n}-2x^n=8$, to find x. *Ans.* $x=\sqrt[n]{2}$.

9. Given $x^{\frac{6}{5}}+x^{\frac{3}{5}}=756$, to find x. *Ans.* $x=243$

10. Given $\frac{8}{(2x-4)^2}=1+\frac{16}{(2x-4)^4}$ to find x. *Ans.* $x=3$ or 1.

11. Given $4x^{\frac{1}{3}}+x^{\frac{1}{6}}=39$, to find x. *Ans.* $x=729$.

12. Given $x^2-2x+6(x^2-2x+5)^{\frac{1}{2}}=11$, to find x.
Ans. $x=1$

13. Given $\frac{x^2}{361}-\frac{12x}{19}=-32$, to find the value of x.
Ans. $x=152$ or 76

If much difficulty is found in resolving this 13th example, the pupil can observe the 9th example, (Art. 99).

14. Given $81x^2+17+\frac{1}{x^2}=99$, to find the values of x.
Ans. $x=1$, or -1, or $\frac{1}{9}$.

Observe that the 1st and 3d terms of the first number are squares, see (Art. 99.)

15. Given $81x^2+17+\frac{1}{x^2}=\frac{841}{x^2}+\frac{232}{x}+15$, to find x.
Ans. $x=2$ or $-1\frac{5}{9}$.

16. Given $25x^2+6+\frac{4}{9x^2}=\frac{955}{9}$, to find the values of x.
Ans. $x=2$, or -2, or $\frac{1}{15}$

17. Given $\frac{4x^2}{49}+\frac{8x}{21}=6\frac{2}{3}$, to find the values of x.

Ans. $x=7$ or $-11\frac{2}{3}$.

(Art. 105.) Equations of the third, fourth, and higher degrees, can be resolved as quadratics, provided we can find a compound quantity in the given equation involved to its *first* and *second* power, with known coefficients.

To determine in any particular case, whether such a compound quantity is involved in the equation, we must transpose all the terms to the first member, and if the highest power of the unknown quantity is not *even*, multiply every term of the equation by the unknown letter to *make it even*, and then extract the square root, to two or three terms, as the case may require; and if we find a remainder to be any multiple or any *aliquot part* of the terms of the root, a reduction to the quadratic form is effected; otherwise it is impossible, and the equation cannot be resolved as a quadratic.

For example, reduce the following equation to the quadratic form, if it be possible.

1. Given $x^4-8ax^3+8a^2x^2+32a^3x-9a^4=0$, to find the values of x by quadratics.

OPERATION.

$$\begin{array}{ll}
 & x^4-8ax^3+8a^2x^2+32a^3x-9a^4=0\ (x^2-4ax) \\
 & \underline{x^4\qquad\qquad} \\
2x^2-4ax & \quad -8ax^3+8a^2x^2 \\
 & \quad \underline{-8ax^3+16a^2x^2\qquad\qquad} \\
 & \qquad\qquad -8a^2x^2+32a^3x-9a^4
\end{array}$$

This remainder can be put into this form:

$$-8a^2(x^2-4ax)-9a^4$$

Now we observe the original equation can be written thus:

$$(x^2-4ax)^2-8a^2(x^2-4ax)-9a^4=0$$

By putting $x^2-4ax=y$ we have

$$y^2-8a^2y=9a^4 \text{ a quadratic.}$$

Completing the square $y^2-8a^2y+16a^4=25a^4$

By evolution $y-4a^2=\pm5a^2$

Hence $y=9a^2 \quad -a^2$

Or $x^2-4ax=9a^2$ or $-a^2$

Completing the square $x^2-4ax+4a^2=13a^2$ or $3a^2$

By evolution $x-2a=\pm a\sqrt{13}$ or $a\sqrt{3}$

Hence x may have the four following values $(2a+a\sqrt{13})$, $(2a-a\sqrt{13})$, $(2a+a\sqrt{3})$, $(2a-a\sqrt{3})$. Either of which being substituted in the original equation will verify it.

2. Reduce $x^3+2ax^2+5a^2x+4a^3=0$ to a quadratic.

As the highest power of x is not *even*, we must multiply by x to make it even. Then

$$x^4+2ax^3+5a^2x^2+4a^3x=0$$

By extracting two terms of the square root, and observing the *remainder*, the part that *will not come into the root*, we find that

$$(x^2+ax)^2+4a^2(x^2+ax)=0$$

Divide by (x^2+ax) and $x^2+ax+4a^2=0$ a quadratic.

3. Given $x^4+2x^3-7x^2-8x+12=0$, to find the values of x.

This equation may be put in the following form:

$$(x^2+x)^2-8(x^2+x)+12=0$$

Ans. $x=1$ or 2, or -2 or -3.

4. Given $x^3-8x^2+19x-12=0$, to find the values of x.

Ans. $x=1$ or 3 or 4.

5. Given $x^4-10x^3+35x^2-50x+24=0$, to find the values of x. *Ans.* $x=1, 2, 3$ or 4.

6. Given $x^4-2x^3+x=132$, to find the values of x.

Ans. $x=4$ or -3.

7. Given $y^4-2cy^3+(c^2-2)y^2+2cy=c^2$, to find the values of y.

Ans. $y=\frac{c}{2}\pm\left(\frac{c^2}{4}+1\pm\sqrt{1+c^2}\right)^{\frac{1}{2}}$.

(Art. 106.) The object of this article is to point out a few little artifices in resolving quadratics, which apply in particular cases only, but which at times may save much labor. It is therefore proper that they should be presented, though some minds prefer uniformity to facility.

For example, take equation (*B*) (Art. 104.)

1. $y^2-5y=6$ Put $2a=5$

Then $y^2-2ay=2a+1$.

Add a^2 to both sides to complete the square, (Rule 1.)

And $y^2-2ay+a^2=a^2+2a+1$

By evolution, $y-a=\pm(a+1.)$

Hence, $y=2a+1=6$ or -1

2. Given $y^2-7y=8$, to find y. *Ans.* $y=8$ or -1

3. Given $x^2+11x=26$, to find the values of x.

Assume $2a=11$; then $4a+4=26$.

Now put these values in place of the numerals, and complete the square, and $x^2+2ax+a^2=a^2+4a+4$.

By evolution, $x+a=\pm(a+2)$ Hence, $x=2$ or -13.

4. Given $x^2-17x=60$, to find the values of x.

Assume $2a=17$; then $6a+9=60$

And $x^2-2ax+a^2=a^2+6a+9$.

By evolution, $x-a=\pm(a+3.)$ Hence, $x=20$ or -3.

5. Given $x^2+19x=92$, to find the values of x.

Assume $2a=19$; then $8a+16=92$

Putting these values and completing the square we have

$$x^2+2ax+a^2=a^2+8a+16$$

$$x+a=\pm(a+4) \quad \text{or} \quad x=4 \text{ or } -23.$$

Observe that in the preceding equations we invariably put the coefficient of the first power of the unknown quantity equal $2a$. Then if we find the absolute term in the second member of the equation equal to $2a+1$

or $4a+4$

or $6a+9$

or $8a+16$

Or, in general, $m2a+m^2$. That is, any multiplier of $2a$ plus the square of the same multiplier equal to the second member, then the equation can be resolved in this manner; for in fact one of the roots of the equation *is* this multiplier of $2a$, and the other root is $\pm(2a+m)$, m being the multiplier, and it may represent any number, integral or fractional; but there is no utility in operating by this method unless m is an integer, and not very large To present a case where m is fractional, we give the following equation: $x^2-9x=\frac{19}{4}$, to find the value of x.

Put $2a=9$; then $\frac{1}{2}\times 2a+\frac{1}{4}=\frac{19}{4}$, and the equation becomes $x^2-2ax=a+\frac{1}{4}$.

Therefore, $x-a=\pm(a+\frac{1}{2})$. Hence, $x=-\frac{1}{2}$ or $2a+\frac{1}{2}=9\frac{1}{2}$.

(Art. 107.) When the roots of the equation are irrational or surd, of course this method of operation will not apply; but we can readily determine whether the roots will be surd or not. For example, take the equation $x^2+13x=40$.

Put $2a=13$; then $4a+4=30$ And $6a+9=48$

From this, we observe that one of the roots of the equation lies between 2 and 3.

(Art. 108.) When the roots of an equation are irrational or surd, no artifice will avail us, and we must conform to set rules; but when the roots are small integers, we can frequently find some method to avoid high numeral quantities; but special artifices can only be taught by examples, *not by precept.* The following are given as examples:

1. Given $x^2+9984x=160000$, to find the values of x.

Observe that $9984=10000-16$

Put $2a=10000$; then $32a=160000$

These substitutions transform the equation to

$$x^2+(2a-16)x=32a$$

Completing the square by (Rule 1) and

$$x^2+(2a-16)x+(a-8)^2=a^2+16a+64$$

By evolution, $x+(a-8)=\pm(a+8)$

Hence, $x=16$ or $-2a=-10000$.

2. Given $x^2+45x=9000$, to find the values of x.

If we put $2a=45$, the multiplier and its square, requisite to produce 9000, is so large that it is not obvious, and of course there will be no advantage in adopting this method; at the same time, we wish to avoid the high numerals we must encounter by any set rule of solution.

We observe that $45\times200=9000$. Put $a=45$

Then $x^2+ax=200a$

Complete the square by (Rule 2,) and

$$4x^2+4ax+a^2=a^2+800a$$

By evolution, $2x+a=\sqrt{a(a+800)}=\sqrt{45\times845}$

Multiply one of the factors, under the radical, by 5, and divide the other by 5, and the equivalent factors will be 225×169, both squares. Taking their root, resuming the value of a, and the equation becomes

$$2x+3.15=13.15$$

Drop 3.15 from both sides

And $2x=10.15$ or $x=75$, *Ans.*

3. Given $16x^2-225x=225$, to find the values of x.

This equation is found in many of the popular works on algebra, and in several of them the common method of resolving it may be seen.

Observe that $225=15\times15$.

Put $a=15$; then $a+1=16$, and the equation becomes

$$(a+1)x^2-a^2x=a^2$$

Completing the square by (Rule 2), and

$$4(a+1)^2x^2-4(a+1)a^2x+a^4=a^4+4a^3+4a^2$$

By evolution, $2(a+1)x-a^2=a^2+2a$

Transpose a^2 and divide by 2, and we have

$$(a+1)x=a^2+a=a(a+1)$$

Divide by $(a+1)$ and $x=a=15$, *Ans*

We give one more example of the utility of representing numerals, or numeral factors, by letters, in reducing the following equation:

4. Given $\frac{18}{x^2}+\frac{81-x^2}{9x}=\frac{x^2-65}{72}$ to find x.

By examining the numerals, we find 9, and several multiples of 9. Therefore, let $a=9$, and using a in the place of 9 the equation becomes

$$\frac{2a}{x^2}+\frac{a^2-x^2}{ax}=\frac{x^2-65}{8a}$$

Clearing of fractions, we have

$$16a^2+8a^2x-8x^3=x^4-65x^2$$

Transposing all to one side, and arranging the terms according to the powers of x, we have

$$\begin{array}{lll} & x^4+8x^3-65x^2-8a^2x-16a^2=0 & (x^2+4x) \\ & x^4 & \\ \hline 2x^2+4x & 8x^3-65x^2 & \\ & 8x^3+16x^2 & \\ \hline & -81x^2-8a^2x-16a^2 & \\ \text{Or} & -a^2(x^2+8x+16) & \end{array}$$

Therefore, by (Art. 105,) the equation becomes

$$(x^2+4x)^2-a^2(x^2+8x+16)=0$$

Or $\quad (x+4)^2x^2=a^2(x+4)^2$

By division, $\quad x^2=a^2$

And $\quad x=\pm a=\pm 9$, *Ans:*

The preceding examples may be of service in reducing some of the following

EXAMPLES.

1. Given $x^2+11x=80$, to find x. *Ans.* $x=5$, or -16

2. Given $5x-\frac{3x-3}{x-3}=2x+\frac{3x-6}{2}$, to find x.

Ans. $x=4$, or -1

3. Given $\frac{x}{x+1}+\frac{x+1}{x}=\frac{13}{6}$, to find x. *Ans.* $x=2$

4. Given $7x+\frac{72x}{10-3x}=50$, to find x. *Ans.* $x=2$, or $\frac{250}{21}$

5. Given $\left(\frac{6}{y}+y\right)^2+\left(\frac{6}{y}+y\right)=30$, to find tne values of y

Ans. $y=3$ or 2, or $-3\pm\sqrt{3}$

6. Given $x^{\frac{4}{3}}+7x^{\frac{2}{3}}=44$, to find the values of x.

Ans. $x=\pm 8$ or $\pm(-11)^{\frac{3}{2}}$.

7. Given $y^2+11+\sqrt{y^2+11}+2=44$, to find the values of y.

Ans. $y=\pm 5$ or $\pm\sqrt{38}$.

8. Given $14+2x-\frac{x+7}{x-7}=x+\frac{9+4x}{3}$, to find the values of x.

Ans. $x=28$ or 9.

9. Given $3x^2-9x-4=80$, to find the values of x.

Ans. $x=7$ or -4.

10. Given $\frac{2\sqrt{x}+2}{4+\sqrt{x}}=\frac{4-\sqrt{x}}{\sqrt{x}}$, to find x. *Ans.* $x=4$.

11. Given $\frac{6(2x-11)}{x-3}+x-2=24-3x$, to find the values of x.

Ans. $x=6$ or $\frac{1}{2}$.

12. Given $\frac{10}{x}-\frac{14-2x}{x^2}=\frac{22}{9}$, to find the values of x.

Ans. $x=3$ or $\frac{21}{11}$.

13. Given $\frac{x^3-10x^2+1}{x^2-6x+9}=x-3$, to find the values of x.

Ans. $x=1$ or -28.

14. Given $mx^2-2mx\sqrt{n}=nx^2-mn$, to find x.

Ans. $x=\frac{\sqrt{mn}}{\sqrt{m}\pm\sqrt{n}}$.

15. Given $x^4+\frac{17x^2}{2}=34x+16$, to find the values of x.

(See Exms. Art. 99.) *Ans* $x=2$, or -2, or -8, or $-\frac{1}{2}$.

16. Given $\left\{\frac{1}{1+x}\left(\frac{1}{1+x}\right)^{\frac{1}{2}}\right\}^{\frac{1}{3}}=\frac{\sqrt{2x}}{12}$, to find the values of x.

N.B. Put $\left(\frac{1}{1+x}\right)=y^2$. *Ans.* $x=8$ or -9.

17. Given $y^4-4y^2=y^2+8y+12$, to find the values of y. (See Art. 99.) *Ans.* $y=3$ or -2.

18. Given $x-1=2+\frac{2}{\sqrt{x}}$, to find the values of x.

Ans. $x=4$ or 1.

19. Given $\left(\frac{x+\sqrt{x^2-9}}{x-\sqrt{x^2-9}}\right)^{\frac{1}{2}}=x-2$, to find the values of x

Ans. $x=5$ or 3.

CHAPTER II.

Quadratic Equations, containing two or more unknown quantities.

(Art. 108.) We have thus far, in quadratics, considered equations involving only one unknown quantity; but we are now fully prepared to carry our investigations farther.

Two equations, essentially quadratic, involving two unknown quantities, depend for their solution on a resulting equation of the fourth degree.

This principle may be shown in the following manner:

Two equations, essentially quadratic, and in the most general form, involving two unknown quantities, may be represented thus:

$$x^2+axy+by^2+cx+dy+e\ \ =0,$$
$$x^2+a'xy+b'y^2+c'x+d'y+e'=0.$$

We do not represent the first terms with a coefficient, as any coefficient may be reduced to unity by division; and a, b, c, &c., and a', b', &c. may represent the result of such division; and, of course, may be of any value, whole or fractional, positive or negative.

Arranging the terms, in the above equations, in reference to x, we have

$$x^2+(ay+c)x+by^2+dy+e\ =0,\quad (1)$$

$$x^2+(a'y+c')x+b'y^2+d'y+e'=0.\quad (2)$$

By subtracting (2) from (1), we have

$$[(a-a')y+c-c']x+(b-b')y^2+(d-d')y+e-e'=0;$$

Therefore $$x=\frac{(b'-b)y^2+(d'-d)y+(e'-e)}{(a-a')y+c-c'}.$$

This expression for x, substituted in either equation (1) or (2) will give a *final* equation, involving only one unknown quantity, y.

But to effect this substitution would lead to a very complicated result; and as our object is only to show the degree to which the resulting equation will rise, we may observe that the expression for the value of x is in the form of $\frac{my^2+ny+q}{ry+s}$. This put in either of the equations (1) or (2), its square, or the expression for x^2, will be of the *fourth* degree; and no term can contain y of a higher degree than the fourth.

Therefore, in general, *the resolution of two equations of the second degree, involving two unknown quantities depends upon that of an equation of the fourth degree involving one unknown quantity.*

(Art. 109.) Two or more equations, involving two or more unknown quantities, can be resolved by quadratics, when they fall under one of the following cases:

1st. One of the equations only may be quadratic; the other must be simple, or capable of being reduced to a simple form.

2d. The equations must be similar in form, or the unknown quantities similarly involved or combined in a similar manner, as they combine in regular powers; or,

3d. The equations must be homogeneous; that is, the exponents of the unknown quantities must make the same sum in every term.

In the first and second cases, we eliminate one quantity in one equation and substitute its value in the other, or perform an equivalent operation, by rules already explained.

In the third case, we throw in a factor to one unknown quantity, to make it equal to the other, or assume it to be so; but these principles can only be explained by

EXAMPLES.

$$ax^2+bxy+cy^2=e,$$
$$a'x^2+b'xy+c'y^2=f.$$

These are homogeneous equations, for the exponents of the unknown quantities make the same sum 2, in every term. In such cases, assume $x=vy$; then the equations become

$$av^2y^2+bvy^2+cy^2=e\,; \quad \text{or} \quad y^2=\frac{e}{av^2+bv+c}$$

$$a'v^2y^2+b'vy^2+c'y^2=f \quad \text{or} \quad y^2=\frac{f}{a'v^2+b'v+c}$$

Hence, $$\frac{e}{av^2+bv+c}=\frac{f}{a'v^2+b'v+c'}$$

An equation involving the 1st and 2d powers of v, and of course, a quadratic.

The solution of this equation will give v. Having v, we have y^2 and y, and from vy we obtain x.

For a particular example, we give the following:

1. Given $\left\{\begin{array}{l} 4x^2-2xy=12 \\ 2y^2+3xy=\ 8 \end{array}\right\}$ to find the values of x and y.

Put $x=vy$; then the equations become

$$4v^2y^2-2vy^2=12, \quad \text{or} \quad y^2=\frac{12}{(2v^2-v)2}$$

And $$2y^2+3vy^2=8, \quad \text{or} \quad y^2=\frac{8}{2+3v}$$

Whence, $\frac{6}{2v^2-v}=\frac{8}{2+3v}$. Dividing by 2, then clearing of fractions, we have

$$6+9v=8v^2-4v \quad \text{or} \quad 8v^2-13v=6.$$

This last equation gives $v=2$ or $-\frac{3}{8}$.

Omitting the negative value $y^2=\frac{8}{2+3v}=\frac{8}{8}=1$.

Therefore, $y=\pm 1$, and $x=vy=\pm 2$.

2. Given $2x-3y=1$, and $2x^2+xy-5y^2=20$, to find the values of x and y.

These equations correspond to the first observation, one of them only being quadratic, the other simple; and the solution is effected by finding the value of x in the first equation. Substituting that value $\frac{3y+1}{2}$ in the 2d, and reducing, we have $2y^2+7y=39$, which gives $y=3$. Hence, $x=5$.

3. Given $x^2+y^2-x-y=78$, and $xy+x+y=39$, to find the values of x and y.

In these equations x and y are similarly involved, not equally involved; nor are the equations homogeneous. In cases of this kind, as we have before remarked, a solution by a quadratic can be effected, but no general or definite rule of operation can be laid down; the hitherto acquired skill of the learner, and his power of comparison to discern the similarity, will do more than any formal rules.

To resolve this example, we multiply the 2d equation by 2, and add the product to the first; we then have

$$x^2+2xy+y^2+x+y=156, \quad \text{or} \quad (x+y)^2+(x+y)=156.$$

Put $x+y=s$; then $s^2+s=156$, a quadratic, which gives s, or $x+y=12$. This value of $x+y$, taken in the second equation, gives $xy=27$. From this sum and product of x and y, we find $x=9$ or 3, and $y=3$ or 9.

Again, after we multiply the second equation by 2, if we subtract it from the first, we shall have

$$\begin{aligned} & x^2-2xy+y^2-3x-3y=0 \\ \text{or} \quad & (x-y)^2-3(x+y)=0 \\ \text{or} \quad & (x-y)^2=3\times 12=36 \\ \text{or} \quad & x-y=\pm 6 \\ \text{But} \quad & x+y=12 \\ \text{Hence,} \quad & 2x=18 \text{ or } 6, \text{ and } x=9 \text{ or } 3, \text{ as before.} \end{aligned}$$

(Art. 110.) There are some equations to which the foregoing observations do not *immediately* apply, or not until after reductions and changes take place. The following is one of them.

4. Given $\left\{\begin{array}{l} x^2+x=\dfrac{12}{y} \\ x^2y+y=18 \end{array}\right\}$ to find the values of x and y.

Here neither of the equations is simple, nor are both letters similarly involved, nor are the equations homogeneous; yet we can find a solution by a quadratic, because the two equations have a common compound factor, which taken away, will bring the equations far within the limits or condition laid down; and this remark will apply to all problems that can be resolved by quadratics not seemingly within the limits of the *three conditions.*

From the first of these equations, we have $y=\dfrac{12}{x^2+x}$

From the second, - - - - $y=\dfrac{18}{x^3+1}$

Hence, $\dfrac{12}{x^2+x}=\dfrac{18}{x^3+1}$. Divide the denominators by $(x+1)$ and the numerators by 6, and we have

$\dfrac{2}{x}=\dfrac{3}{x^2-x+1}$ a quadratic equation.

Clearing of fractions, and $2x^2-2x+2=3x$

or $2x^2-5x=-2.$

(Rule 2.) $16x^2-A+25=25-16=9.$

We write A to represent the second term. It is immaterial what its numeral value may be, as it always disappears in taking the root.

By evolution, $4x-5=\pm 3$

Hence, $x=2$ or $\frac{1}{2}$.

But $y=\dfrac{12}{x^2+x}=\dfrac{12}{4+2}=2$ or $\dfrac{12}{\frac{1}{4}+\frac{1}{2}}=\dfrac{12\times 4}{3}=16$, *Ans.*

The following is of a similar character:

5. Given $\left\{\begin{array}{l} x^2-y^2-(x+y)=\ 8 \\ (x-y)^2(x+y)=32 \end{array}\right\}$ to find the values of x and y.

Divide the first equation by $(x+y)$ and $x-y-1=\frac{8}{x+y}$ (A).

Divide the second by $(x+y)$ and $(x-y)^2=\frac{32}{x+y}$ (B).

Put $x+y=s$, and transpose *minus* 1, in equation (A), and $x-y=\frac{8}{s}+1$. By squaring, $(x-y)^2=\frac{64}{s^2}+\frac{16}{s}+1$.

Equation (B) gives $(x-y)^2=\frac{32}{s}$

Therefore, $\frac{64}{s^2}+\frac{16}{s}+1=\frac{32}{s}$.

Clearing of fractions, and transposing $32s$, we have

$$64-16s+s^2=0$$

By evolution, $8-s=0$ or $s=8$. That is, $x+y=8$, which value, put in equation (A), gives $x-y=2$.

Whence, $x=5$ and $y=3$.

MISCELLANEOUS EXAMPLES.

1. Given $x=2y^2$ and $\frac{1}{3}(x-y)=5$, to find the values of x and y.
Ans. $x=18$ or $12\frac{1}{2}$.
$y=3$ or $-2\frac{1}{2}$.

2. Given $2x+y=22$, and $xy+2y^2=120$, to find the values of x and y. *Ans.* $x=8$, $y=6$.

3. Given $x+y : x-y :: 13 : 5$, and $x+y^2=25$, to find the values of x and y. *Ans.* $x=9$, and $y=4$.

4. Given $\left\{\begin{array}{l} x^3+y^3=18xy \\ x+y\ =12 \end{array}\right\}$ to find the values of x and y.
Ans. $x=8$ or 4, $y=4$ or 8

5. Given $x^2+2xy+y^2=120-2x-2y$, and $xy-y^2=8$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x=6 \text{ or } 9, \text{ or } -9\pm\sqrt{5} \\ y=4 \text{ or } 1, \text{ or } -3\pm\sqrt{5}. \end{array}\right.$

6. Given $\left\{\begin{matrix} x^2+xy=56 \\ xy+2y^2=60 \end{matrix}\right\}$ to find the values of x and y.

Ans. $\left\{\begin{matrix} x=\pm4\sqrt{2} \text{ or } \pm14. \\ y=\pm3\sqrt{2} \text{ or } \pm10. \end{matrix}\right.$

7. Given $\left\{\begin{matrix} 6x^2+2y^2=5xy+12 \\ 2xy+3x^2=3y^2-3 \end{matrix}\right\}$ to find the values of x and y.

Ans. $x=\pm2$, $y=\pm3$

Or $x=\pm\frac{5}{\sqrt{31}}$, $y=\pm\frac{6}{\sqrt{31}}$.

8. Given $\left\{\begin{matrix} 3x^2+xy=68 \\ 4y^2+3xy=160 \end{matrix}\right\}$ to find the values of x and y.

Ans. $x=\pm4$, $y=\pm5$.

In the first four examples, one of the equations is simple; in the 5th and 6th, x and y are similarly involved; and the 6th, 7th and 8th are homogeneous.

(Art. 111.) When we have fractional exponents, we can remove them, as explained in (Art. 92.); but in some cases it may not be important to do so.

EXAMPLES.

1. Given $x^{\frac{3}{2}}+y^{\frac{2}{3}}=3x$ and $x^{\frac{1}{2}}+y^{\frac{1}{3}}=x$, to find the values of x and y.

Put $x^{\frac{1}{2}}=P$; then $x=P^2$ and $x^{\frac{3}{2}}=P^3$

And $y^{\frac{1}{3}}=Q$; then $y=Q^3$ and $y^{\frac{2}{3}}=Q^2$

Now the primitive equations become

$$P^3+Q^2=3P^2, \text{ and } P+Q=P^2$$

From the 1st, $Q^2=(3-P)P^2$

From the 2d, $Q=(P-1)P$

By squaring, $Q^2=(P-1)^2P^2$

Put the two values of Q^2 equal to each other, rejecting or dividing by the common factor P^2, and we have

$$(P-1)^2=3-P$$

or $P^2-2P+1=3-P$

or $P^2-P=2$

Hence, $P=2$ or 1, and $x=4$ or 1, and $y=8$.

2. Given $x^{\frac{2}{3}}+y^{\frac{2}{3}}+2x^{\frac{1}{3}}+2y^{\frac{1}{3}}=23$, and $x^{\frac{2}{3}}y^{\frac{2}{3}}=36$, to find the values of x and y. *Ans.* $x=27$ or 8, $y=8$ or 27.

3. Given $x^{\frac{1}{3}}+y^{\frac{1}{3}}=8$, and $x^{\frac{2}{3}}+y^{\frac{2}{3}}+x^{\frac{2}{3}}y^{\frac{2}{3}}=259$, to find the values of x and y.

Ans. $\begin{cases} x=125 \text{ or } 27, \\ y=27 \text{ or } 125. \end{cases}$

4. Given $x^{\frac{4}{3}}+y^{\frac{2}{5}}+x^{\frac{2}{3}}+y^{\frac{1}{5}}=26$, and $x^{\frac{2}{3}}y^{\frac{1}{5}}=8$, to find the values of x and y. *Ans.* $x=8$, $y=32$.

5. Given $\frac{x}{y}+\frac{4x^{\frac{1}{2}}}{y^{\frac{1}{2}}}=\frac{33}{4}$ and $x-y=5$, to find the values of x and y. *Ans.* $x=9$, $y=4$

6. Given $\left\{\begin{array}{l} y^2-8x^{\frac{1}{2}}y=64 \\ y-2x^{\frac{1}{2}}y^{\frac{1}{2}}=4 \end{array}\right\}$ to find the values of x and y *Ans.* $x=2\frac{1}{4}$, $y=16$

7. Given $x(y^{\frac{1}{2}}+1)+y(x^{\frac{1}{2}}+1)+2x^{\frac{1}{2}}y^{\frac{1}{2}}=55$, and $xy^{\frac{1}{2}}+yx^{\frac{1}{2}}=30$, to find the values of x and y. *Ans.* $\begin{cases} x=4 \text{ or } 9 \\ y=9 \text{ or } 4 \end{cases}$

8. Given $x^{\frac{2}{3}}y^{\frac{3}{2}}=2y^2$ and $8x^{\frac{1}{3}}-y^{\frac{1}{2}}=14$, to find the values of x and y. *Ans.* $\begin{cases} x=2744 \text{ or } 8 \\ y=9604 \text{ or } 4 \end{cases}$

(Art. 112.) No additional principles, to those already given, are requisite for the solution of problems containing three or more unknown quantities in quadratics. As in simple equations, we must have as many independent equations as unknown quantities.

As auxiliary to the solution of certain problems, particularly in geometrical progression, we give the following problem:

Given $x+y=s$, *and* $xy=p$, *to find the values of* x^2+y^2, x^3+y^3, x^4+y^4, *and* x^5+y^5, *expressed in terms of* s *and* p.

Squaring the first, $x^2+2xy+y^2=s^2$

Subtract twice the second, $2xy \quad =2p$

1st result, $x^2+y^2 \quad =s^2-2p$ (A)

Again,

$(x+y)(x^2+y^2)=x^3+x^2y+xy^2+y^3=s^3-2ps$

Subtract $xy(x+y) \quad = \quad ps$

2d result, $x^3+y^3=s^3-3ps$ (B)

Square (A), and $x^4+2x^2y^2+y^4=s^4-4s^2p+4p^2$

Subtract $2x^2y^2 \quad = \quad 2p^2$

3d result, $x^4+y^4=s^4-4s^2p+2p^2$ (C)

Multiply (A) by (B) and

$x^5+x^3y^2+x^2y^3+y^5=s^5-5s^3p+6sp^2$

Subtract $x^2y^2(x+y) \quad = \quad sp^2$

4th result, $x^5+y^5 \quad =s^5-5s^3p+5sp^2$ (D)

CHAPTER III.

Questions producing Quadratic Equations.

(Art. 113.) The method of proceeding to reduce the question into equations, is the same as in simple equations ; and, in fact, many problems which result in a quadratic may be brought out by simple equations, by foresight and skill in notation. Others again are so essentially quadratic, that no expedient can change their form.

EXAMPLES.

1. A person bought a number of sheep for $240. If there had been 8 more, they would have coſt him $1 a-piece less. What was the cost of a sheep, and how many did he purchase?

Let $x=$ the number of sheep; then $\frac{240}{x}=$cost of one.

If he had $x+8$ sheep, $\frac{240}{x+8}$=cost of one.

By the question, $\frac{240}{x}=\frac{240}{x+8}+1$

Clearing of fractions, $240x+1920=240x+x^2+8x$

Or $x^2+8x=1920$

Resolving gives $x=40$ or -48; but a minus number will not apply to sheep; the other value only will apply to the problem as enunciated.

This question can be brought into a simple equation thus: Let $x-4=$ the number of sheep, then 8 more would be expressed by $x+4$, and the equation would be

$$\frac{240}{x-4}=\frac{240}{x+4}+1. \quad \text{Put } a=240.$$

Then $\frac{a}{x-4}=\frac{a}{x+4}+1$

Clearing of fractions, $ax+4a=ax-4a+x^2-16.$

Transposing, $x^2=8a+16=8(a+2)=16\times 121$

Extracting square root, $x=4\times 11=44$. Hence, $x-4=40$, the number of sheep. Divide 240 by 40, and we have $6 for the price of one sheep.

(Art. 114.) In resolving problems, if the second member is negative after completing the square, it indicates some impossibility in the conditions from which the equation is derived, or an error in forming the equation, and in such cases the values of the unknown quantity are both *imaginary*.

2. For example, let it be required to divide 20 into two such parts that their product shall be 140.

Let $x=$ one part, then $20-x=$ the other.

By conditions, $20x-x^2=140$

Or, $x^2-20x=-140$

Completing the square, $x^2-20x+100=-40$

By evolution, $x-10=\pm 2\sqrt{-10}$

Or, $x=10 \pm 2\sqrt{-10}$

This result shows an impossibility; there are no such parts of 20 as here expressed. It is impossible to divide 20 into two such parts that their product shall be over 100, the product of 10 by 10, and so on with any other number. The product of two parts is the greatest possible, when the parts are equal.

3. Find two numbers, such that the sum of their squares being subtracted from three times their product, 11 will remain; and the difference of their squares being subtracted from twice their product, the remainder will be 14.

Let $x=$ the greater number, and $y=$the less.

By the conditions, $3xy-x^2-y^2=11$

And $2xy-x^2+y^2=14$

These are homogeneous equations; therefore, put $x=vy$;

Then $3vy^2-v^2y^2-y^2=11 \qquad (A)$

And $2vy^2-v^2y^2+y^2=14 \qquad (B)$

Conceive (A) divided by (B) and the fraction reduced, we have

$$\frac{3v-v^2-1}{2v-v^2+1}=\frac{11}{14}$$

Clearing of fractions and reducing, we find

$$3v^2-20v=-25.$$

A solution gives one value of v, $\frac{5}{3}$

Put this value in equation (A), and we have

$$5y^2-\frac{25y^2}{9}-y^2=11$$

Multiply by 9, and $45y^2-25y^2-9y^2=11\times9$,

Or, $11y^2=11\times9$,

$y^2=9$ or $y=3$. Hence, $x=5$.

4. A company dining at a house of entertainment, had to pay $3.50; but before the bill was presented two of them went away; in consequence of which, those who remained had to pay each 20 cents more than if all had been present. How many persons dined? *Ans.* 7.

5. There is a certain number, which being subtracted from 22, and the remainder multiplied by the number, the product will be 117. What is the number? *Ans.* 13 or 9.

6. In a certain number of hours a man traveled 36 miles, but if he had traveled one mile more per hour, he would have taken 3 hours less than he did to perform his journey. How many miles did he travel per hour? *Ans.* 3 miles.

7. A person dies, leaving children and a fortune of $46,800, which, by the will, is to be divided equally among them; but it happens that immediately after the death of the father, two of the children also die; and if, in consequence of this, each remaining child receive $1950 more than he or she was entitled to by the will, how many children were there? *Ans.* 8 children.

8. A gentleman bought a number of pieces of cloth for 675 dollars, which he sold again at 48 dollars by the piece, and gained by the bargain as much as one piece cost him. What was the number of pieces? *Ans.* 15.

This problem produces one of the equations in (Art. 10.)

9. A merchant sends for a piece of goods and pays a certain sum for it, besides 4 per cent. for carriage; he sells it for $390, and thus gains as much per cent. on the cost and carriage as the 12th part of the purchase money amounted to. For how much did he buy it? *Ans.* $300.

10. Divide the number 60 into two such parts that their produce shall be 704. *Ans.* 44 and 16.

11. A merchant sold a piece of cloth for $39, and gained as much per cent. as it cost him. What did he pay for it? *Ans.* $30.

12. *A* and *B* distributed 1200 dollars each, among a certain number of persons. *A* relieved 40 persons more than *B*, and *B* gave to each individual 5 dollars more than *A*. How many were relieved by *A* and *B*? *Ans.* 120 by *A*, and 80 by *B*.

This problem can be brought into a pure equation, in like manner as (Problem 1.)

13. A vintner sold 7 dozen of sherry and 12 dozen of claret for £50, and finds that he has sold 3 dozen *more* of sherry for £10 than he has of claret for £6. Required the price of each?

Ans. Sherry, £2 per dozen; claret, £3.

14. *A* set out from *C* towards *D*, and traveled 7 miles a day. After he had gone 32 miles, *B* set out from *D* towards *C*, and went every day $\frac{1}{19}$ of the whole journey; and after he had traveled as many days as he went miles in a day, he met *A*. Required the distance from *C* to *D*.

Ans. 76 or 152 miles; both numbers will answer the condition.

15. A farmer received $24 for a certain quantity of wheat, and an equal sum at a price 25 cents less by the bushel for a quantity of barley, which exceeded the quantity of wheat by 16 bushels. How many bushels were there of each?

Ans. 32 bushels of wheat, and 48 of barley.

16. *A* and *B* hired a pasture, into which *A* put 4 horses, and *B* as many as cost him 18 shillings a week; afterwards *B* put in two additional horses, and found that he must pay 20 shillings a week. At what rate was the pasture hired?

Ans. *B* had six horses in the pasture at first, and the price of the whole pasture was 30 shillings per week.

17. Find those two numbers whose sum, product, and difference of their squares, are all equal to each other.

Ans. $\frac{1}{2}(3\pm\sqrt{5})$, and $\frac{1}{2}(1\pm\sqrt{5})$.

18. If a certain number be divided by the product of its two digits, the quotient will be 2, and if 27 be added to the number, the digits will be inverted. What is the number?

Ans. 36.

19. It is required to find three numbers, whose sum is 33, such that the difference of the first and second shall exceed the difference of the second and third by 6, and the sum of whose squares is 441. *Ans.* 4, 13, and 16.

20. Find those two numeral quantities whose sum, product, and sum of their squares, are all equal to each other.

Ans. No such numeral quantities exist. In a strictly algebraic sense, the quantities are

$$\tfrac{3}{2}\pm\tfrac{1}{2}\sqrt{-3}, \text{ and } \tfrac{3}{2}\mp\tfrac{1}{2}\sqrt{-3}.$$

21. What two numbers are those whose product is 24, and whose sum added to the sum of their squares is 62?

Ans. 4 and 6.

22. It is required to find two numbers, such that if their product be added to their sum it shall make 47, and if their sum be taken from the sum of their squares, the remainder shall be 62?

Ans. 7 and 5.

23. The sum of two numbers is 27, and the sum of their cubes 5103. What are their numbers? *Ans.* 12 and 15

24. The sum of two numbers is 9, and the sum of their fourth powers 2417. What are the numbers? *Ans.* 7 and 2.

25. The product of two numbers multiplied by the sum of their squares, is 1248, and the difference of their squares is 20 What are the numbers? *Ans.* 6 and 4.

Let $x+y$=the greater, and $x-y$=the less.

26. Two men are employed to do a piece of work, which they can finish in 12 days. In how many days could each do the work alone, provided it would take one 10 days longer than the other? *Ans.* 20 and 30 days.

27. The joint stock of two partners, *A* and *B*, was $1000. *A*'s money was in trade 9 months, and *B*'s 6 months; when they shared stock and gain, *A* received $1,140 and *B* $640. What was each man's stock?

Ans. *A*'s stock was $600; *B*'s $400.

28. A speculator from market, going out to buy cattle, met with four droves. In the second were 4 more than 4 times the square root of one half the number in the first. The third contained three times as many as the first and second. The fourth was one half the number in the third and 10 more, and the whole number in the four droves was 1121. How many were in each drove? *Ans.* 1st, 162; 2d, 40; 3d, 606; 4th, 313

29. Divide the number 20 into two such parts, that the product of their squares shall be 9216. *Ans.* 12 and 8.

30. Divide the number *a* into two such parts that the product of their squares shall be *b*.

Ans. Greater part $\frac{a}{2}+\frac{1}{2}\left(a^2-4\sqrt{b}\right)^{\frac{1}{2}}$.

Less part $\frac{a}{2}-\frac{1}{2}\left(a^2-4\sqrt{b}\right)^{\frac{1}{2}}$.

31. Find two numbers, such that their product shall be equal to the difference of their squares, and the sum of their squares shall be equal to the difference of their cubes.

Ans. $\pm\frac{1}{2}\sqrt{5}$ and $\frac{1}{4}(5\pm\sqrt{5})$

SECTION V.

ARITHMETICAL PROGRESSION.

CHAPTER I.

A series of numbers or quantities, increasing or decreasing by the same difference, from term to term, is called arithmetical progression.

Thus, 2, 4, 6, 8, 10, 12, &c., is an increasing or ascending arithmetical series, having a common difference of 2; and 20, 17, 14, 11, 8, &c., is a decreasing series, having a common difference of 3.

(Art. 15.) We can more readily investigate the properties of an arithmetical series from literal than from numeral terms. Thus let *a* represent the first term of a series, and *d* the common difference. Then

$a, (a+d), (a+2d), (a+3d), (a+4d)$, &c., represent an ascending series; and

$a, (a-d), (a-2d), (a-3d), (a-4d)$, &c., represent a descending series.

Observe that the coefficient of d, in any term is equal to the number of the preceding terms.

The first term exists without the common difference. All other terms consist of the first term and the common difference multiplied by *one* less than the number of terms from the *first.*

Wherever the series is supposed to terminate, is the last term, and if such term be designated by L, and the number of terms by n, the last term must be $a+(n-1)d$, or $a-(n-1)d$, according as the series may be ascending or descending, which we draw from inspection.

$$\text{Hence, } L=a\pm(n-1)d \qquad (A)$$

(Art. 116.) It is manifest that the sum of the terms will be the same, in whatever order they are written.

Take, for instance, the series	3,	5,	7,	9,	11,
And the same inverted,	11,	9,	7,	5,	3.
The sums of the terms will be	14,	14,	14,	14,	14.

Take the series	a,	$a+d$,	$a+2d$,	$a+3d$,	$a+4d$,
Inverted,	$a+4d$,	$a+3d$,	$a+2d$,	$a+d$,	a
Sums will be	$2a+4d$,	$2a+4d$,	$2a+4d$,	$2a+4d$,	$2a+4d$.

Here we discover the important property, that, in an arithmetical progression, *the sum of the extremes is equal to the sum of any other two terms equally distant from the extremes. Also, that twice the sum of any series is equal to the extremes, or first and last term repeated as many times as the series contains terms.*

Hence, if S represents the sum of a series, and n the number of terms, a the first term, and L the last term, we shall have

$$2S=n(a+L)$$

$$\text{Or} \qquad S=\frac{n}{2}(a+L) \qquad (B)$$

The two equations (A) and (B) contain *five* quantities, a, d,

L, n, and S; any three of them being given, the other *two* can be determined.

Two independent equations are sufficient to determine two unknown quantities, (Art. 45,) and it is immaterial which two are unknown if the other three are given.

By examining the two equations

$$L=a+(n-1)d \qquad (A)$$

$$S=\frac{n}{2}(a+L) \qquad (B)$$

We perceive that the value of any letter, L for example, can be drawn from equation (B) as well as from (A).

It can also be drawn from either of the equations after n or a is eliminated from them. Hence, the value of L may take *four* different forms. The same may be said of the other letters, and there being five quantities or letters and four different forms to each, the subject of arithmetical progression *may include twenty* different equations. But we prefer to make no display with these equations, believing they would add darkness rather than light, as they are all essentially included in the two equations, (A) and (B), and these can be remembered literally and philosophically, and the entire subject more surely understood.

These two equations are sufficient for problems relating to arithmetical series, and we may use them without modification by putting in the given values just as they stand, and afterwards reducing them as numeral equations.

EXAMPLES.

1 The sum of an arithmetical series is 1455, the first term 5, and the number of terms 30. What is the common difference?

Ans. 3.

Here $S=1455$, $a=5$, $n=30$. L and d are sought.

Equation (B) $1455=(5+L)15$. Reduced $L=92$

Equation (A) $92=5+29d$. Reduced $d=3$, *Ans.*

2. The sum of an arithmetical series is 567, the first term 7, and the common difference 2. What is the number of terms?

Ans. 21

Here $s=567$, $a=7$, $d=2$. L and n are sought.

Equation (A) $L=7+2n-2=5+2n$

Equation (B) $567=(7+5+2n)\frac{n}{2}=6n+n^2$

Or $n^2+6n+9=576$

$n+3=24$, or $n=21$, *Ans.*

3. Find *seven* arithmetical means between 1 and 49.

Observe that the series must consist of 9 terms.

Hence, $a=1$, $L=49$, $n=9$.

Ans. 7, 13, 19, 25, 31, 37, 43.

4. The first term of an arithmetical series is 1, the sum of the terms 280, the number of terms 32. What is the common difference, and the last term? *Ans.* $d=\frac{1}{2}$, $L=16\frac{1}{2}$.

5. Insert three arithmetical means between $\frac{1}{3}$ and $\frac{1}{2}$.

Ans. The means are $\frac{3}{8}$, $\frac{5}{12}$, $\frac{11}{24}$

6. Find nine arithmetical means between 9 and 109.

Ans. $d=10$.

7. What debt can be discharged in a year by paying 1 cent the first day, 3 cents the second, 5 cents the third, and so on, increasing the payment each day by 2 cents?

Ans. 1332 dollars 25 cents.

8. A footman travels the first day 20 miles, 23 the second, 26 the third, and so on, increasing the distance each day 3 miles. How many days must he travel at this rate to go 438 miles?

Ans. 12.

9. What is the sum of n terms of the progression of 1, 2, 3, 4, 5, &c.?

Ans. $S=\frac{n}{2}(1+n)$.

10. The sum of the terms of an arithmetical series is 950, the common difference is 3, and the number of terms 25. What is the first term? *Ans.* 2.

11. A man bought a certain number of acres of land, paying for the first, \$$\frac{1}{3}$; for the second, \$$\frac{2}{3}$; and so on. When he came to settle he had to pay \$3775. How many acres did he purchase, and what did it average per acre?

Ans. 150 acres at \$$25\frac{1}{6}$ per acre

Problems in Arithmetical Progression to which the preceding formulas, (A) and (B), do not immediately apply.

(Art. 117.) When three quantities are in arithmetical progression, it is evident that the middle one must be the exact *mean* of the three, otherwise it would not be arithmetical progression; therefore the sum of the extremes must be double of the mean.

Take, for example, any three consecutive terms of a series, as

$$a+2d, \quad a+3d, \quad a+4d;$$

and we perceive by inspection that the sum of the extremes is double the mean.

When there are four terms, the sum of the extremes is equal to the sum of the means, by (Art. 116.)

To facilitate the solution of problems, when three terms are in question, let them be represented by $(x-y)$, x, $(x+y)$, y being the common difference.

When four numbers are in question, let them be represented by $(x-3y)$, $(x-y)$, $(x+y)$, $(x+3y)$; $2y$ being the common difference.

So in general for any other number, assume such terms *that the common difference will disappear by addition.*

1. There are five numbers in arithmetical progression, the sum of these numbers is 65, and the sum of their squares is 1005. What are the numbers?

Let $x=$ the middle term, and y the common difference. Then $x-2y$, $x-y$, x, $x+y$, $x+2y$, will represent the numbers, and their sum will be $5x=65$, or $x=13$. Also, the sum of their squares will be

$$5x^2+10y^2=1005 \quad \text{or} \quad x^2+2y^2=201.$$

But $x^2=169$; therefore, $2y^2=32$, $y^2=16$ or $y=4$.

Hence, the numbers are $13-8=5$, 9, 13, 17 and 21

2. There are three numbers in arithmetical progression, their sum is 18, and the sum of their squares 158. What are those numbers? *Ans.* 1, 6 and 11

3. It is required to find four numbers in arithmetical progression, the common difference of which shall be 4, and their continued product 176985. *Ans.* 15, 19, 23 and 27

4. There are four numbers in arithmetical progression, the sum of the extremes is 8, and the product of the means 15 What are the numbers? *Ans.* 1, 3, 5, 7.

5. A person travels from a certain place, goes 1 mile the first day, 2 the second, 3 the third, and so on; and in six days after, another sets out from the same place to overtake him, and travels uniformly 15 miles a day. How many days must elapse after the second starts before they come together?

Ans. 3 days and 14 days

Reconcile these two answers.

6. A man borrowed \$60; what sum shall he pay daily to cancel the debt, principal and interest, in 60 days; interest at 10 per cent. for 12 months, of 30 days each?

Ans. \$1 and $\frac{61}{72}$ of a cent.

7. There are four numbers in arithmetical progression, the sum of the squares of the extremes is 50, the sum of the squares of means is 34; what are the numbers? *Ans.* 1, 3, 5, 7

8. The sum of four numbers in arithmetical progression is 24, their continued product is 945. What are the numbers?

Ans. 3, 5, 7, 9.

9. A certain number consists of three digits, which are in arithmetical progression, and the number divided by the sum of its digits is equal to 26; but if 198 be added to the number its digits will be inverted What is the number? *Ans.* 234

CHAPTER II.

GEOMETRICAL PROGRESSION.

(Art. 118.) When numbers or quantities differ from each other by a constant multiplier in regular succession, they constitute a geometrical series, and if the multiplier be greater than unity, the series is ascending; if it be less than unity, the series is descending.

Thus, 2 : 6 : 18 : 54 : 162 : 486, is an ascending series, the multiplier, called the ratio, being three ; and 81 : 27 : 9 : 3 : 1 : $\frac{1}{3}$: $\frac{1}{9}$, &c., is a descending series, the multiplier or ratio being $\frac{1}{3}$.

Hence, $a : ar : ar^2 : ar^3 : ar^4 : ar^5 : ar^6$: &c., may represent any geometrical series, and if r be greater than 1, the series is ascending, if less than 1, it is descending.

(Art. 119.) Observe that the *first* power of r stands in the 2d term, the 2d power in the 3d term, the *third* power in the 4th term, and thus universally *the power of the ratio* in any term is one less than the number of the term.

The first term is a factor in every term. Hence the 10th term of this general series is ar^9. The 17th term would be ar^{16}. The nth term would be ar^{n-1}.

Therefore, if n represent the number of terms in any series, and L the last term, then $\qquad L=ar^{n-1} \qquad (1)$

(Art. 120.) If we represent the sum of any geometrical series by s, we have

$$s=a+ar+ar^2+ar^3+ \text{ \&c.} \ldots ar^{n-2}+ar^{n-1}.$$

Multiply this equation by r, and we have

$$rs=ar+ar^2+ar^3+ \text{ \&c. } ar^{n-1}+ar^n.$$

Subtract the upper from the lower, and observe that

$$Lr=ar^n; \text{ then } (r-1)s=Lr-a.$$

Therefore, $$s=\frac{Lr-a}{r-1}. \qquad (2)$$

As these two equations are fundamental, and cover the whole subject of geometrical progression, let them be brought together for critical inspection.

$$L=ar^{n-1} \quad (1), \qquad S=\frac{Lr-a}{r-1} \quad (2).$$

These two equations furnish the rules given for the operations in common arithmetic.

Here we perceive five quantities, a, r, n, L and S, and any three of them being given in any problem, the other two can be determined from the equations.

To these equations we may apply the same remarks as were made to the two equations in arithmetical progression (Art. 116.)

Equation (2), put in words, gives the following rule for the sum of a geometrical series;

Rule. *Multiply the last term by the ratio, and from the product subtract the first term, and divide the remainder by the ratio less one.*

EXAMPLES FOR THE APPLICATION OF EQUATIONS (1) AND (2).

1. Required the sum of 9 terms of the series, 1, 2, 4, 8, 16, &c. *Ans.* 511.

2. Required the 8th term of the progression, 2, 6, 18, 54, &c. *Ans.* 4374.

3. What is the sum of ten terms of the series 1, $\frac{2}{3}$, $\frac{4}{9}$, &c.? *Ans.* $\frac{174075}{59049}$

4. Required two geometrical means between 24 and 192.

N. B. When the two means are found, the series will consist of four terms; the first term 24 and the last term 192.

By equation (1) $L=ar^{n-1}$.

Here $a=24$, $L=192$, $n=4$, and the equation becomes

$$192=24r^3 \text{ or } r=2.$$

Hence, 48 and 96 are the means required.

5. Required 7 geometrical means between 3 and 768.
Ans. 6, 12, 24, 48, 96, 192.

6. The first term of a geometrical series is 5, the last term 1215, and the number of terms 6. What is the ratio? *Ans.* 3.

7. A man purchased a house, giving $1 for the first door, $2 for the second, $4 for the third, and so on, there being 10 doors. What did the house cost him? *Ans.* $1023.

(Art. 121.) By Equation (2), and the Rule subsequently given, we perceive that the sum of a series depends on the first and last terms and the ratio, *and not on the number of terms;* and whether the terms be many or few, there is no variation in the rule. Hence, we may require the sum of any descending series, as 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c., to infinity, *provided we determine the* LAST *term.* Now we perceive the magnitude of the terms decrease as the series advances; the hundredth term would be extremely small, the thousandth term very much less, and the *infinite term* nothing; not too small to be noted, as some tell us, but absolutely *nothing.*

Hence, in any decreasing series, when the number of terms is *conceived* to be infinite, the last term, L, becomes 0, and Equation (2) becomes $s=\frac{-a}{r-1}$.

By change of signs $s=\frac{a}{1-r}$.

This gives the following rule for the sum of a decreasing infinite series:

Rule. *Divide the first term by the difference between unity and the ratio.*

EXAMPLES.

1. Find the value of 1, $\frac{3}{4}$, $\frac{9}{16}$, &c., to infinity.

$a=1$, $r=\frac{3}{4}$. *Ans.* 4

2. Find the exact value of the decimal .3333, &c., to infinity. *Ans.* $\frac{1}{3}$.

This may be expressed thus: $\frac{3}{10}+\frac{3}{100}$, &c. Hence, $a=\frac{3}{10}$, $r=\frac{1}{10}$

3. Find the value of .323232, &c., to infinity.

$a=\frac{32}{100}$, $ar=\frac{32}{10000}$; therefore $r=\frac{1}{100}$. *Ans.* $\frac{32}{99}$

4. Find the value of .777, &c., to infinity. *Ans.* $\frac{7}{9}$.

5. Find the value of $\frac{5}{3} : 1 : \frac{3}{5} : \frac{9}{25}$, &c. to infinity.
Ans. $4\frac{1}{6}$

6. Find the value of $5 : \frac{5}{3} : \frac{5}{9}$, &c. to infinity. *Ans.* $7\frac{1}{2}$

7. Find the value of the series $\frac{1}{5}$, $\frac{1}{25}$, &c., to infinity? *Ans.* $\frac{1}{4}$.

8. What is the value of the decimal .71333, &c., to infinity?
Ans. $\frac{214}{300}$.

9. What is the value of the decimal .212121, &c., to infinity?
Ans. $\frac{7}{33}$.

(Art. 122.) If we observe the general series, (Art. 118.) $a : ar : ar^2 : ar^3 : ar^4$, &c., we shall find, by taking three consecutive terms anywhere along in the series, that *the product of the extremes will equal the square of the mean.* Hence, to find a geometrical mean between two numbers, we must multiply them together, and take the square root. *If we take four consecutive terms, the product of the extremes will be equal to the product of the means.*

(Art. 123.) This last property belongs equally to geometrical proportion, as well as to a geometrical series, and the learner must be careful not to confound *proportion* with a *series.*

$a : ar :: b : br$, is a geometrical proportion, *not* a continued series. The ratio is the same in the two couplets, but the magnitudes a and b, to which the ratio is applied, may be very different.

We may suppose $a : ar$ two consecutive terms of one series, and $b : br$ any two consecutive terms of another series having the same ratio as the first series, and being brought together they form a geometrical proportion. *Hence, the equality of the ratio constitutes proportion.*

To facilitate the solution of some difficult problems in geometrical progression, it is desirable, if possible, to express several terms by two letters only, and have them *stand symmetrically.*

Three terms may be expressed by $x : \sqrt{xy} : y$, or by $x^2 : xy : y^2$, as the product of the extremes is here evidently equal to the square of the mean.

To express four terms with x and y symmetrically, we at first write $P : x :: y : Q$. The first three being in geometrical progression, gives $Py = x^2$ or $P = \frac{x^2}{y}$. In the same manner, we find $Q = \frac{y^2}{x}$ And taking these values of P and Q we have $\frac{x^2}{y} : x :: y : \frac{y^2}{x}$ to represent four numbers symmetrically with two letters.

Taking three numbers as above, and placing them between P and Q, thus, $P : x^2 : xy : y^2 : Q$, we have five numbers; and by reducing P and Q into functions of x and y, we have $\frac{x^3}{y} : x^2 : xy : y^2 : \frac{y^3}{x}$, for five terms symmetrically expressed.

Six numbers thus, $\frac{x^3}{y^2} : \frac{x^2}{y} : x :: y : \frac{y^2}{x} : \frac{y^3}{x^2}$

Sometimes we may more advantageously express unknown numbers in geometrical proportion by x, xy, xy^2, &c.; x being the first term, and y the ratio.

HARMONICAL PROPORTION.

(Art. 124.) When three magnitudes, a, b, c, have the relation of $a : c :: a-b : b-c$; that is, the first is to the third as the difference between the first and second is to the difference between the second and third, the quantities a, b, c, are said to be in *harmonical proportion.*

(Art. 125.) Four magnitudes are in harmonical proportion when the first is to the fourth as the difference between the first and second is to the difference between the third and fourth. Thus, a, b, c, d, are in harmonical proportion when $a : d :: a-b : c-d$, or when $a : d :: b-a : d-c$.

An harmonical mean between two numbers is equal to twice their product divided by their sum. For $a : x : b$ representing three numbers in harmonical proportion, we have by the definition, (Art. 124.) $a : b :: a-x : x-b$.

Therefore, $ax-ab=ab-bx$ or $x=\frac{2ab}{a+b}$.

1. Find the harmonical mean between 6 and 12. *Ans.* 8.

2. Find the third harmonical number to 234 and 144.
Ans. 104.

3. Find the fourth harmonical proportion to the numbers 24 : 16 : 4. *Ans.* 3.

4. There are four numbers in harmonical proportion, the first is 16, the third 3, the fourth 2. The second is lost; find it.
Ans. 8

PROBLEMS IN GEOMETRICAL PROGRESSION, AND HARMONICAL PROPORTION.

1. The sum of three numbers in geometrical progression is 26, and the sum of their squares 364. What are the numbers?

Let the numbers be represented by $x : \sqrt{xy} : y$.

Then $x+\sqrt{xy}+y = 26=a \qquad (1)$

And $x^2+ xy+y^2=364=b \qquad (2)$

Transpose, $\sqrt{xy}$ in Equation (1) and square, we have

$$x^2+2xy+y^2=a^2-2a\sqrt{xy}+xy \qquad (3)$$

Or $x^2+ xy+y^2=a^2-2a\sqrt{xy} \qquad (4)$

The left hand members of Equations (2) and (4) are equal; hence,

$$a^2-2a\sqrt{xy}=b.$$

Therefore,
$$\sqrt{xy}=\frac{a^2-b}{2a}=6 \qquad (5)$$

This gives the second term of the progression, and now from equations (1) and (5) we find $x=2$, $y=18$, and the numbers are 2, 6, 18.

2. The sum of four numbers in geometrical progression is 15, (a), and the sum of their squares 85 (b). What are the numbers?

Let the numbers be represented as in (Art. 123.)

$$\left.\begin{aligned}&\text{Then}\quad \frac{x^2}{y}+x+y+\frac{y^2}{x}=a \quad (1)\\ &\text{And}\quad \frac{x^4}{y^2}+x^2+y^2+\frac{y^4}{x^2}=b \quad (2)\end{aligned}\right\}$$

Assume $x+y=s$

$xy=p$

Then by (Art. 112.)

$x^2+y^2=s^2-2p$

And $x^3+y^3=s^3-3sp$

Transposing $(x+y)$ in Equation (1), and (x^2+y^2) in Equation (2), we have

$$\frac{x^2}{y}+\frac{y^2}{x}=a-s \;(3) \text{ and } \frac{x^4}{y^2}+\frac{y^4}{x^2}=b-s^2+2p \qquad (4)$$

Square (3) and transpose $2xy$ or $2p$ and

$$\frac{x^4}{y^2}+\frac{y^4}{x^2}=(a-s)^2-2p \qquad (5)$$

The left hand members of equations (4) and (5) are equal, therefore,

$$(a-s)^2-2p=b-s^2+2p$$

Or $$a^2-2as+2s^2-4p=b \qquad (6)$$

Clear equation (3) of fractions, and $x^3+y^3=ap-ps$.

That is, $s^3-3sp=ap-ps$ or $p=\dfrac{s^3}{a+2s}$ (7)

Put this value of p in equation (6) and reduce, we have,

$$a^2-2as^2=ab+2bs$$

Or $$as^2+bs=\frac{a}{2}(a^2-b)$$

Taking the given values of a and b we have

$15s^2+85s=70\times15$

Or $3s^2+17s=210$, an equation which gives $s=6$

Put the values of a and s in equation (7), and $p=8$

That is, $x+y=6$, and $xy=8$, from which we find $x=2$, and $y=4$; therefore the required numbers are

1, 2, 4 and 8, *Ans*

3. The arithmetical mean of two numbers exceeds the geo

metrical mean by 13, and the geometrical mean exceeds the harmonical mean by 12. What are the numbers?

Let x and y represent the numbers.

Then $\frac{1}{2}(x+y)=$ the arithmetical mean, $\sqrt{xy}=$ the geometrical mean, (Art. 112.) and $\frac{2xy}{x+y}=$ the harmonical mean.

Let $a=12$;

Then, by the question, $\frac{1}{2}(x+y)=\sqrt{xy}+a+1$ (1)

And $\sqrt{xy}=\frac{2xy}{x+y}+a$ (2)

By our customary substitution, these equations become

$$\tfrac{1}{2}s=\sqrt{p}+a+1 \qquad (3)$$

And $\sqrt{p}=\frac{2p}{s}+a$ (4)

Take the value of s from equation (3) and put it into equation (4), dividing the numerator and the denominator by 2, and we have

$$\sqrt{p}=\frac{p}{\sqrt{p}+a+1}+a \qquad (5)$$

Clearing of fractions, we shall have

$$p+a\sqrt{p}+\sqrt{p}=p+a\sqrt{p}+(a+1)a$$

Drop equals, and $\sqrt{p}=(a+1)a$ (6)

Put this value of $\sqrt{p}$ in equation (3) and we have

$$\tfrac{1}{2}s=(a+1)a+(a+1)=(a+1)(a+1)$$

Or $s=2(a+1)^2$ (7)

For the sake of brevity, put $(a+1)=b$; squaring equation (6) and restoring the values of s and p in equations (6) and (7), and we have

$$xy=a^2b^2 \qquad (A)$$

$$x+y=2b^2 \qquad (B)$$

Square (B) and

$$x^2+2xy+y^2=4b^4$$

Subtract 4 times (A) $\Big\}$ $4xy=4a^2b^2$

And $x^2-2xy+y^2=4b^2(b^2-a^2)=4b^2(b+a)(b-a)$ (C)

As $a=12$ and $b=13$, $b+a=25$, and $b-a=1$.

Therefore, (C) becomes $(x-y)^2=4b^2\times25\times1$.

By evolution, $x-y=2b\times5$

Equation (B) $x+y=2b^2$

By addition $2x=2b^2+10b$

Or $x=b^2+5b=(b+5)b=18\times13=234$

By subtraction, $2y=2b^2-10b$

$y=b^2-5b=(b-5)b=8\times13=104$.

A more brief solution is the following:

Let $x-y$ and $x+y$ represent the numbers.

Then $x=$ the arithmetical mean, $\sqrt{x^2-y^2}=$ the geometrical mean, (Art. 112), and $\frac{x^2-y^2}{x}=$ the harmonical mean. By the question,

$$x-13=\sqrt{x^2-y^2}\ (1),\text{ and }\frac{x^2-y^2}{x}+12=\sqrt{x^2-y^2}\qquad(2)$$

The right hand members of equations (1) and (2) being the same, therefore, $\frac{x^2-y^2}{x}+12=x-13$.

By reduction, $y^2=25x$.

Put this value of y^2 in equation (1), and by squaring

$x^2-26x+(13)^2=x^2-25x$, or $x=(13)^2=169$.

Hence, $y=65$, and the numbers are 104 and 234.

4. Divide the number 210 into three parts, so that the last shall exceed the first by 90, and the parts be in geometrical progression. *Ans.* 30, 60, and 120.

5. The sum of four numbers in geometrical progression is 30; and the last term divided by the sum of the mean terms is $1\frac{1}{3}$. What are the numbers? *Ans.* 2, 4, 8, and 16.

6. The sum of the first and third of four numbers in geometrical progression is 148, and the sum of the second and fourth is 888. What are the numbers?

Ans. 4, 24, 144, and 864.

7. It is required to find three numbers in geometrical progression, such that their sum shall be 14, and the sum of their squares 84. *Ans.* 2, 4, and 8.

8. There are four numbers in geometrical progression, the second of which is less than the fourth by 24; and the sum of the extremes is to the sum of the means, as 7 to 3. What are the numbers? *Ans.* 1, 3, 9 and 27.

9. The sum of four numbers in geometrical progression is equal to the common ratio $+1$, and the first term is $\frac{1}{10}$. What are the numbers? *Ans.* $\frac{1}{10}$, $\frac{3}{10}$, $\frac{9}{10}$, $\frac{27}{10}$.

10. The sum of three numbers in harmonical proportion is 26, and the product of the first and third is 72. What are the numbers? *Ans.* 12, 8, and 6

11. The continued product of three numbers in geometrical progression is 216, and the sum of the squares of the extremes is 328. What are the numbers? *Ans.* 2, 6, 18.

12. The sum of three numbers in geometrical progression is 13, and the sum of the extremes being multiplied by the mean, the product is 30. What are the numbers?

Ans. 1, 3, and 9

13. There are three numbers in harmonical proportion, the sum of the first and third is 18, and the product of the three is 576. What are the numbers? *Ans.* 6, 8, 12.

14. There are three numbers in geometrical progression, the difference of whose difference is 6, and their sum 42. What are the numbers? *Ans.* 6, 12, 24.

15. There are three numbers in harmonical proportion, the difference of whose difference is 2, and three times the product of the first and third is 216. What are the numbers?

Ans. 6, 8, and 12.

16. Divide 120 dollars between four persons, in such a way, that their shares may be in arithmetical progression; and if the second and third each receive 12 dollars less, and the fourth 24 dollars more, the shares would then be in geometrical progression. Required each share.

Ans. Their shares were 3, 21, 39, and 57, respectively.

17. There are three numbers in geometrical progression, whose sum is 31, and the sum of the first and last is 26. What are the numbers? *Ans.* 1, 5, and 25.

18 The sum of six numbers in geometrical progression is 189 and the sum of the second and fifth is 54. What are the numbers? *Ans.* 3, 6, 12, 24, 48 and 96.

19. The sum of six numbers in geometrical progression is 189, and the sum of the two means is 36. What are the numbers? *Ans.* 3, 6, 12, 24, 48 and 96.

CHAPTER III.

PROPORTION.

(Art. 176.) We have given the definition of geometrical proportion in (Art. 41.) and demonstrated the most essential property, the equality of the products between extremes and means. We now propose to extend our investigations a little farther.

Proportion can only exist between magnitudes of the same kind, and the number of times and parts of a time, that one measures another, is called the *ratio*. Ratio is always a number, and *not a quantity*.

(Theorem 1.) *If two magnitudes have the same ratio as two others, the first two as numerator and denominator may form one member of an equation; and the other two magnitudes as numerator and denominator will form the other member.*

Let A and B represent the first two magnitudes and r their ratio. Also C and D the other two magnitudes, and r their ratio.

Then, $\frac{A}{B}=r$ and $\frac{C}{D}=r$ Therefore, (Ax. 7) $\frac{A}{B}=\frac{C}{D}$

(Theorem 2.) *Magnitudes which are proportional to the same proportionals, are proportional to each other.*

$$\left.\begin{array}{ll}\text{Suppose} & a:b::P:Q\\ \text{and} & c:d::P:Q\\ \text{and} & x:y::P:Q\end{array}\right\}\quad\begin{array}{c}\text{Then we are to prove that}\\ a:b::c:d\\ \text{and } a:b::x:y, \text{\&c.}\end{array}$$

From the first proportion, $\frac{b}{a}=\frac{Q}{P}$

From the second, $\frac{d}{c}=\frac{Q}{P}$

Hence, (Ax. 7) $\frac{b}{a}=\frac{d}{c}$ or $a:b::c:d$

In the same manner we prove $a:b::x:y$

And $c:d::x:y$

(Theorem 3.) *If four magnitudes constitute a proportion, the first will be to the sum of the first and second, as the third is to the sum of the third and fourth.*

By hypothesis, $a:b::c:d$; then we are to prove that $a:a+b::c:c+d$.

By the given proportion, $\frac{b}{a}=\frac{d}{c}$. Add unity to both members, then reducing to the form of a fraction we have $\frac{b+a}{a}=\frac{d+c}{c}$. Throwing this equation into its equivalent proportional form, we have,

$$a:a+b::c:c+d.$$

N. B. In place of adding unity, subtract it, and we shall find that

$$a:a-b::c:c-d.$$

or $$a:b-a::c:d-c.$$

(Theorem 4.) *If four magnitudes be proportional, the sum of the first and second is to their difference, as the sum of the third and fourth is to their difference.*

Admitting that $a:b::c:d$, we are to prove that

$$a+b:a-b::c+d:c-d$$

From the same hypothesis, (Theorem 3.) gives

$$a:a+b::c:c+d$$

And $$a:a-b::c:c-d$$

Changing the means, (which will not affect the product of the extremes and means, and of course will not destroy proportionality,) and we have,

$$a : c :: a+b : c+d$$
$$a : c :: a-b : c-d$$

Now by (Theorem 2.) $a+b : c+d :: a-b : c-d$

Changing the means, $a+b : a-b :: c+d : c-d$

(Theorem 5.) *If four magnitudes be proportional, like powers or roots of the same, will be proportional.*

Admitting $a : b :: c : d$, we are to show that

$$a^n : b^n :: c^n : d^n, \text{ and } a^{\frac{1}{n}} : b^{\frac{1}{n}} :: c^{\frac{1}{n}} : d^{\frac{1}{n}}$$

By the hypothesis, $\frac{a}{b}=\frac{c}{d}$. Raising both members of this equation to the nth power, and

$$\frac{a^n}{b^n}=\frac{c^n}{d^n}$$

Changing this to the proportion $a^n : b^n :: c^n : d^n$

Resuming again the equation $\frac{a}{b}=\frac{c}{d}$, and taking the nth root of each member, we have $\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}=\frac{c^{\frac{1}{n}}}{d^{\frac{1}{n}}}$. Converting this equation into its equivalent proportion, we have

$$a^{\frac{1}{n}} : b^{\frac{1}{n}} :: c^{\frac{1}{n}} : d^{\frac{1}{n}}.$$

Now by the first part of this theorem, we have

$a^{\frac{m}{n}} : b^{\frac{m}{n}} :: c^{\frac{m}{n}} : d^{\frac{m}{n}}$, m representing any power whatever, and n representing any root.

(Theorem 6.) *If four magnitudes be proportional, also four others, their compound, or product of term by term, will form a proportion.*

Admitting that $a : b :: c . d$

And $x : y :: m : n$

We are to show that, $ax : by :: mc : nd$

From the first proportion, $\frac{a}{b}=\frac{c}{d}$.

From the second, $\frac{x}{y}=\frac{m}{n}$

Multiply these equations, member by member, and

$$\frac{ax}{by}=\frac{mc}{nd}$$

Or $ax : by :: mc : nd$.

The same would be true in any number of proportions.

(Theorem 7.) *Taking the same hypothesis as in* (Theorem 6.) *we propose to show, that a proportion may be formed by dividing one proportion by the other, term by term.*

By hypothesis, $a : b :: c : d$

And $x : y :: m : n$

Multiply extremes and means, $ad=bc$ (1)

And $nx=my$ (2)

Divide (1) by (2), and $\frac{a}{x}\times\frac{d}{n}=\frac{c}{m}\times\frac{b}{y}$

Convert these four terms, which make two equal products, into a proportion, and we shall have

$$\frac{a}{x} : \frac{b}{y} :: \frac{c}{m} : \frac{d}{n}$$

By comparing this with the given proportions, we find it composed of the quotients of the several terms of the first proportion divided by the corresponding term of the second.

(Theorem 8.) *If four magnitudes be proportional, we may multiply the first couplet or the second couplet, the antecedents or the consequents, or divide them by the same factor, and the results will be proportional in every case.*

Suppose . . . . $a : b :: c : d$

Multiply ex. and means, and . . . . $ad=bc$ (1)

Multiply this eq. by m, and . . . . $mad=mbc$

Now, in this last equation, ma may be considered as a single term or factor, or md may be so considered. So, in the second member, we may take mb as one factor, or mc. Hence we may convert this equation into a proportion in four different ways.

Thus, as . . . . $ma : mb :: c : d$

or as . . . . $a : b :: mc : md$

or as . . . . $ma : b :: mc : d$

or as . . . . $a : mb :: c : md$

If we resume the original equation (1), and divide it by any number, m, in place of multiplying it, we can have, by the same course of reasoning,

$$\frac{a}{m} : \frac{b}{m} :: c : d$$

$$a : b :: \frac{c}{m} : \frac{d}{m}$$

$$\frac{a}{m} : b :: \frac{c}{m} : d$$

$$a : \frac{b}{m} :: c : \frac{d}{m}$$

The following examples are intended to illustrate the practical utility of the foregoing theorems:

EXAMPLES.

1. Find two numbers, the greater of which shall be to the less, as their sum to 42; and as their difference is to 6.

Let x=the greater, y=the less.

Then, per question, $\begin{cases} x : y :: x+y : 42 & (1) \\ x : y :: x-y : 6 & (2) \end{cases}$

(Theorem 2.) $x+y : 42 :: x-y : 6$

Changing the means $x+y : x-y :: 42 : 6$

(Theorem 4.) $2x : 2y :: 48 : 36$

(Art. 42.) $x : y :: 4 : 3$

(Theorem 2.) $4 : 3 :: x-y : 6$

And $4 : 3 :: x+y : 42$

From these last proportions, $x-y=8$

And $x+y=56$. Hence, $x=32$, $y=24$.

2. Divide the number 14 into two such parts, that the quotient of the greater divided by the less shall be to the quotient of the less divided by the greater, as 16 to 9.

Let $x=$ the greater part, and $14-x=$the less.

By the conditions, $\frac{x}{14-x} : \frac{14-x}{x} :: 16 : 9$

Multiplying terms, $x^2 : (14-x)^2 :: 16 : 9$

Extracting root, $x : (14-x) :: 4 : 3$ (Theor. 5.)

Adding terms, $x : 14 :: 4 : 7$

Dividing terms, $x : 2 :: 4 : 1$

Therefore, $x=8$.

3. There are three numbers in geometrical progression whose sum is 52, and the sum of the extremes is to the mean as 10 to 3. What are the numbers? *Ans.* 4, 12, and 36.

Let x, xy, xy^2 represent the numbers.

Then, by the conditions, $x+xy+xy^2=52$ (1)

And $xy^2+x : xy :: 10 : 3$

(Art. 42.) $y^2+1 : y :: 10 : 3$

Double 2d and 4th, $y^2+1 : 2y :: 10 : 6$

Adding and sub. terms, $y^2+2y+1 : y^2-2y+1 :: 16 : 4$

Extracting square root, $y+1 : y-1 :: 4 : 2$

Adding and sub. terms, $y : 1 :: 3 : 1$. Hence, $y=3$.

From equation (1), $x=\frac{52}{1+y+y^2}=\frac{52}{1+3+9}=\frac{52}{13}=4$.

4. The product of two numbers is 35, the difference of their cubes, is to the cube of their difference as 109 to 4. What are the numbers? *Ans.* 7 and 5

Let x and y represent the numbers.

Then, by the conditions, $xy=35$, and $x^3-y^3 : (x-y)^3 :: 109:4$

Divide by $(x-y)$ (Art. 42.) and $x^2+xy+y^2 : (x-y)^2 :: 109:4$

Expanding, and $x^2+xy+y^2 : x^2-2xy+y^2 :: 109:4$

(Theorem 3.) $3xy : (x-y)^2 :: 105:4$

But $3xy$, we know from the first equation, is equal to 105.

Therefore, $(x-y)^2=4$, or $x-y=2$.

We can obtain a very good solution of this problem by putting $x+y=$ the greater, and $x-y=$ the less of the two numbers.

5. What two numbers are those, whose difference is to their sum as 2 to 9, and whose sum is to their product as 18 to 77? *Ans.* 11 and 7

6. Two numbers have such a relation to each other, that if 4 be added to each, they will be in proportion as 3 to 4; and if 4 be subtracted from each, they will be to each other as 1 to 4 What are the numbers? *Ans.* 5 and 8

7. Divide the number 16 into two such parts that their product shall be to the sum of their squares as 15 to 34. *Ans.* 10, and 6.

8. In a mixture of rum and brandy, the difference between the quantities of each, is to the quantity of brandy, as 100 is to the number of gallons of rum; and the same difference is to the quantity of rum, as 4 to the number of gallons of brandy. How many gallons are there of each? *Ans.* 25 of rum, and 5 of brandy.

9. There are two numbers whose product is 320; and the difference of their cubes, is to the cube of their difference, as 61 to 1. What are the numbers? *Ans.* 20 and 16.

10. Divide 60 into two such parts, that their product shall be to the sum of their squares as 2 to 5. *Ans.* 40 and 20

11. There are two numbers which are to each other as 3 to 2. If 6 be added to the greater and subtracted from the less, the sum and the remainder will be to each other, as 3 to 1. What are the numbers? *Ans.* 24 and 16.

12. There are two numbers, which are to each other, as 16 to 9, and 24 is a mean proportional between them. What are the numbers? *Ans.* 32 and 18.

13. The sum of two numbers is to their difference as 4 to 1, and the sum of their squares is to the greater as 102 to 5. What are the numbers? *Ans.* 15 and 9.

14. If the number 20 be divided into two parts, which are to each other in the *duplicate* ratio of 3 to 1, what number is a mean proportional between those parts?

Ans. 18 and 2 are the parts, and 6 is the mean proportion between them.

15. There are two numbers in proportion of 3 to 2; and if 6 be added to the greater, and subtracted from the less, the results will be as 3 to 1. What are the numbers? *Ans.* 24 and 16.

16. There are three numbers in geometrical progression, the product of the first and second, is to the product of the second and third, as the first is to twice the second; and the sum of the first and third is 300. What are the numbers?

Ans. 60, 120, and 240.

17. The sum of the cubes of two numbers, is to the difference of their cubes, as 559 to 127; and the square of the first, multiplied by the second, is equal to 294. What are the numbers? *Ans.* 7 and 6.

18. There are two numbers, the cube of the first is to the square of the second, as 3 to 1; and the cube of the second is to the square of the first as 96 to 1. What are the numbers?

Ans. 12 and 24.

SECTION VI.

CHAPTER I.

INVESTIGATION AND GENERAL APPLICATION OF THE BINOMIAL THEOREM.

(Art. 127.) It may seem natural to continue right on to the higher order of equations, but in the resolution of some cases in cubics, we require the aid of the binomial theorem; it is therefore requisite to investigate that subject now.

The just celebrity of this theorem, and its great utility in the higher branches of analysis, induce the author to give a general demonstration: and the pupil cannot be urged too strongly to give it special attention.

In (Art. 67.) we have expanded a binomial to several powers by actual multiplication, and in that case, derived a law for forming exponents and coefficients when the power was a whole positive number; but the great value and importance of the theorem arises from the fact that the general law drawn from that case is equally true, when the exponent is fractional or negative, and therefore it enables us to extract *roots*, as well as to expand *powers*.

(Art. 128.) Preparatory to our investigation, we must prove the truth of the following theorem:

If there be two series arising from different modes of expanding the same, or equal quantities, with a varying quantity having regular powers in each series; then the coefficients of the same powers of the varying quantity in the two series are equal.

For example, suppose

$$A+Bx+Cx^2+Dx^3, \text{ \&c. } =a+bx+cx^2+dx^3, \text{ \&c.}$$

This equation is true by hypothesis, through all values of x. It is true then, when $x=0$. Make this supposition, and $A=a$ Now let these equal values be taken away, and the remainder divided by x. Then again, suppose $x=0$, and we shall find $B=b$. In the same manner we find $C=c$, $D=d$, $E=e$, &c.

(Art. 129.) A binomial in the form of $a+x$ may be put in the form of $a\times\left(1+\frac{x}{a}\right)$; for we have only to perform the multiplication here indicated to obtain $a+x$. Hence $(a+x)^m=a^m\left(1+\frac{x}{a}\right)^m$.

Now if we can expand $\left(1+\frac{x}{a}\right)^m$, it will be sufficient to multiply every term of the expanded series by a^m for the expansion of $(a+x)^m$, but as every power or root of 1 is 1, the first term of the expansion of $\left(1+\frac{x}{a}\right)^m$ is 1, and this multiplied by a^m must give a^m for the first term of the expansion of $(a+x)^m$, whatever m may be, *positive or negative, whole or fractional.*

As we may put x in place of $\frac{x}{a}$, we perceive that any binomial may be reduced to the form of $(1+x)$, which, for greater facility, we shall operate upon.

(Art. 130.) Let it be required to expand $(1+x)^m$, when m is a positive whole number. By actual multiplication, it can be shown, as in (Art. 67.) that the first term will be 1, and the second term mx. For if $m=2$, then

$$(1+x)^m=(1+x)^2=1+2x, \text{ \&c.}$$

If $m=3$, $(1+x)^m=1+3x$, &c.

And in general, $(1+x)^m=1+mx+Ax^2,+Bx^3$, &c.

The exponent of x increasing by unity every term, and A, B, C, &c., unknown coefficients, which have some law of dependence on the exponent m, which it is the object of this investigation to discover.

(Art. 131.) Now if m is supposed to be a fraction, or if $m=\frac{1}{r}$, the expansion of $(1+x)^m$ will be a root in place of a power, and we must expand $(1+x)^{\frac{1}{r}}$.

For example, let us suppose $r=2$, then $(1+x)^{\frac{1}{r}}=(1+x)^{\frac{1}{2}}$, and to examine the *form* the series would take, let us actually undertake to extract the square root of $(1+x)$ by the common rule.

OPERATION.

$$\begin{array}{lll} & & 1+x\left(1+\frac{1}{2}x-\frac{1}{2.4}x^2\right. \\ 1 & & \underline{1} \\ 2+\frac{1}{2}x & & x \\ & & \underline{x+\frac{1}{4}x^2} \\ 2+x & & -\frac{1}{4}x^2 \end{array}$$

Thus we perceive that in case of square root, the first term of the series must be unity, and the coefficient of the second term is the *index* of the binomial, and the powers of x increase by unity from term to term.

We should find the same laws to govern the *form* of the series, if we attempted to extract cube, or any other root; but, to be *general* and scientific, we must return to the literal expression $(1+x)^{\frac{1}{r}}$.

Now as any root of 1 is 1, the first term of this root must be 1; and the second term will have some coefficient to x. *Let that coefficient be represented by* p; and as the powers of r will increase by unity every term, we shall have

$$(1+x)^{\frac{1}{r}}=1+px+Ax^2. \quad \&c.$$

Take the r power of both members, and we shall have

$$1+x=(1+px+Ax^2, \quad \&c.)^r$$

As r is a whole number, we can expand this second member by multiplication; that is, by (Art. 130.), the second member must take the following form

$$1+x=1+rpx+A'x^2, \quad \&c.$$

Drop 1, and divide by x, and we have

$$1=rp+A'x+, \quad \&c.$$

By (Art. 128.) $\quad 1=rp \quad$ or $\quad p=\frac{1}{r}$;

That is, the coefficient represented by p must be equal to the index of the binomial.

Therefore $(1+x)^{\frac{1}{r}}=1+\frac{1}{r}x+Ax^2+Bx^3+$, &c.; the same general form as when the exponent was considered a whole number.

(Art. 132,) If we take $m=\frac{n}{r}$ we have to expand the root of a power. The first term must be 1, and the second term will contain x, with some coefficient, and the coefficients of x will rise higher and higher every term.

That is, $(1+x)^{\frac{n}{r}}=1+px+Ax^2$, &c. Take the r power of both members, and $(1+x)^n=(1+px,\ \text{&c.})^r$.

Expanding both members, as in (Art. 130),

$$1+nx+ax^2,\ \text{&c.}=1+rpx+Ax^2,\ \text{&c.}$$

Now, by (Art. 128), $n=rp$ or $p=\frac{n}{r}$.

Therefore, $(1+x)^{\frac{n}{r}}=1+\frac{n}{r}x+Ax^2+Bx^3$, &c.; the first two terms following the same law, relative to the exponents, as in the former cases. Now let us suppose m negative. Then

$$(1+x)^{\overset{-}{m}} \text{ will become } (1+x)^{-m}=\frac{1}{(1+x)^m} \text{ (Art. 18.)}$$

$$\text{Or by (Art. 130.)} \quad \frac{1}{1+mx+Ax^2,\ \text{&c.}}$$

By actual division, $1+mx+Ax^2$, &c.) 1 $(1-mx+A'x^2$, &c.

$$\begin{array}{l} \underline{1+mx+Ax^2} \\ -mx-Ax^2 \\ -mx-m^2x^2 \end{array}$$

That is, $(1+x)^{-m}=1-mx+A'x^2$, which shows that the same general law governs the coefficient of the *second term*, as in the former cases.

Hence it appears that whether the exponent m of a binomial be *positive* or *negative*, *whole* or *fractional*, the same general form of expression must be preserved.

That is, in all cases $(1+x)^m=1+mx+Ax^2+Bx^3$, &c.

(Art 133.) For clearness of conception, let the pupil bear in mind that the coefficients of an expanded binomial quantity *depend not at all* on the magnitudes of the quantities themselves, but on the *exponent*. Thus, $(a+b)$ to the 5th, or to any other power, the coefficients will be the same, whether a and b are great or small quantities, or whatever be their relation to each other.

(Art. 134.) The equation $(1+x)^m=1+mx+Ax^2+Bx^3$, &c., is supposed to be true, therefore it must be true, if we square both members. But we have only a portion of one member. We have, however, as much as we please to assume, and sufficient to determine the leading terms of its square, which is all that we desire. Square both members, and $(1+2x+x^2)^m=(1+mx+Ax^2+Bx^3+Cx^4, \text{\&c.})^2$.

By expanding the second member, and arranging the terms according to the powers of x, we shall have

$$(1+2x+x^2)^m=1+2mx+\left.\begin{matrix}m^2\\2A\end{matrix}\right|x^2+\left.\begin{matrix}2B\\2mA\end{matrix}\right|x^3+\left.\begin{matrix}2C\\2mB\\A^2\end{matrix}\right|x^4 \quad \text{\&c.} \qquad (1)$$

Now if we assume $y=2x+x^2$, the first member of this equation becomes $(1+y)^m$. If we expand this binomial into a series, it must have the same coefficients as the expansion of $(1+x)^m$, because the coefficients depend entirely on the exponent m, (Art. 133.)

Therefore, $(1+y)^m=1+my+Ay^2+By^3+Cy^4$, &c.

Put the values of y, y^2, y^3, &c., in this equation, and arrange the terms according to the powers of x, and we have

$$(1+2x+x^2)^m=1+2mx+\left.\begin{matrix}m\\4A\end{matrix}\right|x^2+\left.\begin{matrix}4A\\8B\end{matrix}\right|x^3+\left.\begin{matrix}A\\12B\\16C\end{matrix}\right|x^4, \text{\&c.} \qquad (2)$$

The left hand members of equations (1) and (2), are identical, and the coefficients of like powers in the second member must be equal. (Art. 128.)

Therefore, $4A+m=m^2+2A$, or $A=\frac{m^2-m}{2}=m\frac{m-1}{2}$

And $8B+4A=2mA+2B$

$$\text{Therefore } B=\frac{A(m-2)}{3}=m.\frac{m-1}{2}.\frac{m-2}{3} \qquad (3)$$

By putting the coefficients of the fourth powers of x equal, we have

$$14C+12B+A=2mB+A^2$$

To obtain the value of C from this equation, in the requisite form, is somewhat difficult.

We must make free use of the preceding values of A and B, which are alone sufficient; but, to facilitate the operation, we shall make use of the following auxiliary.

Assume $P=m-2$, then $P+1=m-1$,

$$\text{and} \qquad \frac{mP+m}{2}=m.\frac{m-1}{2}=A \qquad (a)$$

Also, by inspecting equation (3), we perceive that

$$3B=AP \qquad \text{and} \qquad 2mB=\frac{2mAP}{3} \qquad (b)$$

By putting the values of $12B$ and $2mB$ in the primitive equation, and dividing every term by A, and in the second member taking the value of A from equation (a), and we shall have

$$\frac{14C}{A}+4P+1=\frac{2mP}{3}+\frac{mP+m}{2}$$

Multiply by 6, and substitute the value of

$24P+6=24m-42$, because $P=m-2$, and we have

$$\frac{84C}{A}+24m-42=4mP+3mP+3m.$$

Collecting terms, and dividing by 7, gives

$$\frac{12C}{A}+3m-6=mP$$

But $3m-6=3P$, which substitute and transpose, and

$$\frac{12C}{A}=mP-3P=P(m-3).$$

Or $$C=\frac{AP(m-3)}{12}=m\,\frac{m-1}{2}\cdot\frac{m-2}{3}\cdot\frac{m-3}{4}$$

Therefore the development of $(1+x)^m$ must be

$$1+mx+m\cdot\frac{m-1}{2}x^2+m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}x^3+$$

$$m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}\cdot\frac{m-3}{4}x^4+\ \&c.$$

whatever be the value of m, positive or negative, whole or fractional, and thus far the law of development has *been demonstrated*, and we infer, and can only infer, that this law will continue true, whatever be the number of terms. Hence, the demonstration is complete, only so far as we extend it.

This series will terminate when (m) is a whole positive number, but in all other cases it will be infinite.

Few pupils, who pay attention, find difficulty in comprehending the preceding method of demonstration, and for that reason we preferred it; but to the following demonstration, no objection as to completeness or perfection can be found.

SECOND DEMONSTRATION.

Assume $(1+x)^m=1+Ax+Bx^2+Cx^3+Dx^4+$ &c. (1)

Then $(1+y)^m=1+Ay+By^2+Cy^3+Dy^4+$ &c. (2)

By subtracting (2) from (1) and dividing the result by $(x-y)$ we obtain

$$\frac{(1+x)^m-(1+y)^m}{x-y}=A+\frac{B(x^2-y^2)}{x-y}+\frac{C\,(x^3-y^3)}{x-y}+\frac{D(x^4-y^4)}{x-y}$$
$$+\ \&c. \quad (3)$$

Place $(1+x)=P$, and $(1+y)=Q$. (a)

Then $x-y=P-Q$. Whence (3) becomes

$$\frac{P^m-Q^m}{P-Q}=A+B\left(\frac{x^2-y^2}{x-y}\right)+C\left(\frac{x^3-y^3}{x-y}\right)+D\left(\frac{x^4-y^4}{x-y}\right)+\ \&c.$$

If we divide (x^2-y^2) by $(x-y)$ the quotient will terminate with the second term, and (x^3-y^3) divided by $(x-y)$ the quotient will terminate with its third term, and so on.

Then if n is a whole number, the division of the first member as indicated will terminate with the n^{th} term of the quotient.

Now, divide in the last equation as indicated, and afterward suppose $x=y$. It then follows that $P=Q$, and we have $P^{n-1}+P^{n-2}Q+P^{n-3}Q^2+$ &c. to n terms. And if $P=Q$ the whole quotient must be nP^{n-1} for the first member.

Whence, $nP^{n-1}=A+2Bx+3Cx^2+4Dx^3+5Ex^4$ &c.

But $P \quad =1+x.$

Multiply these two equations, member by member, and we shall have,

$$nP^n=A+\left.\begin{matrix}2B\\+A\end{matrix}\right|x+\left.\begin{matrix}3C\\2B\end{matrix}\right|x^2+\left.\begin{matrix}4D\\3C\end{matrix}\right|x^3+\left.\begin{matrix}3E\\4D\end{matrix}\right|x^4+\text{ \&c.}$$

But we perceive, by inspecting equation (1), that

$$nP^n=n+nAx+nBx^2+nCx^3+mDx^4+\text{ \&c.}$$

By equating the right hand members of these two equations, and observing that the coefficients of the like powers of x, must be equal, we shall have

$$A=n$$

$$2B+A=nA \qquad \text{or} \qquad B=A\left(\frac{n-1}{2}\right)$$

$$3C+2B=nB \qquad \text{or} \qquad C=B\left(\frac{n-2}{3}\right)$$

$$4D+3C=nC \qquad \text{or} \qquad D=C\left(\frac{n-3}{3}\right)$$

$$\text{\&c.} \qquad\qquad \text{\&c.}$$

Whence

$$A=n$$

$$B=n.\frac{n-1}{2}$$

$$C=n.\frac{n-1}{2}.\frac{n-2}{3}$$

$$D=n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}$$

and so on, the law of continuation being obvious from the equations, to whatever term it may be extended.

Thus the demonstration is complete when n is a whole positive number.

Now let n be an irreducible fraction, as $\frac{s}{t}$,

Then $(1+x)^{\frac{s}{t}}=1+Ax+Bx^2+Cx^3+Dx^4$ $\qquad(a)$

And $(1+y)^{\frac{s}{t}}=1+Ay+By^2+Cy^3+Dy^4$ $\qquad(b)$

Now assume $(1+x)^{\frac{1}{t}}=P$ and $(1+y)^{\frac{1}{t}}=Q$

Then, $(1+x)^{\frac{s}{t}}=P^s$ " $(1+y)^{\frac{s}{t}}=Q^s$
And $1+x=P^t$ $\qquad 1+y=Q^t$
Whence $x-y=P^t-Q^t$ $\qquad(c)$

Now if we subtract (b) from (a), and divide the remainder by $x-y$, we shall have

$$\frac{P^s-Q^s}{P^t-Q^t}=A+B\left(\frac{x^2-y^2}{x-y}\right)+C\left(\frac{x^3-y^3}{x-y}\right)+D\left(\frac{x^4-y^4}{x-y}\right)+ \text{ \&c.} \qquad (d)$$

Now divide numerator and denominator of the first member by $(P-Q)$, and it will become

$$\frac{\;\frac{P^s-Q^s}{P-Q}\;}{\;\frac{P^t-Q^t}{P-Q}\;}=\frac{P^{s-1}+P^{s-2}Q+,\text{ \&c.}}{P^{t-1}+P^{t-2}Q+,\text{ \&c.}}$$

Now, because s and t are whole numbers, the division in both numerator and denominator will be complete, and if we suppose $x=y$, it follows that $P=Q$, and the results of the several divisions will be

$$\frac{sP^{s-1}}{tQ^{t-1}}=\frac{s}{t}.\,P^{s-t}=A+2Bx+3Cx^2+4Dx^3+,\text{ \&c.} \qquad (e)$$

But $\qquad P^t=1+x.$

Multiplying these equations, member by member, will produce

$$\frac{s}{t}P^s=A\begin{array}{l}+2B\\+A\end{array}\Bigg|\,x+\begin{array}{l}3C\\2B\end{array}\Bigg|\,x^2+\begin{array}{l}4D\\3C\end{array}\Bigg|\,x^3+,\text{ \&c.} \qquad (f)$$

But by inspecting equation (a), and observing that $(1+x)^{\frac{s}{t}}=P^s$

$$\frac{s}{t}P^{s}=\frac{s}{t}+\frac{s}{t}Ax+\frac{s}{t}Bx^{2}+\frac{s}{t}Cx^{3}+,\ \&c. \qquad (g)$$

Equating (f) and (g), and operating as before, we find

$$A=\frac{s}{t}$$

$$B=A\frac{\frac{s}{t}-1}{2}=\frac{s}{t}\cdot\frac{\frac{s}{t}-1}{2}$$

$$C=B\frac{\frac{s}{t}-2}{3}=\frac{s}{t}\cdot\frac{\frac{s}{t}-1}{2}\cdot\frac{\frac{s}{t}-2}{3},$$

$$\&c. \qquad \&c.$$

The same law of development as before, and this completes the demonstration for all positive fractional exponents.

Now, let the exponent be a negative fraction,

Then, $(1+x)^{-\frac{s}{t}}=1+Ax+Bx^{2}+Cx^{3}+,\ \&c. \qquad (a')$

$(1+y)^{-\frac{s}{t}}=1+Ay+By^{2}+Cy^{3}+,\ \&c. \qquad (b')$

Now assume, $(1+x)^{-\frac{s}{t}}=P^{-s}$ Then, $(1+x)^{-\frac{1}{t}}=P^{-t}$

Or, $P^{t}=1+x$

In like manner, $Q^{t}=1+y$

By subtraction, $P^{t}-Q^{t}=x-y$

Subtracting (b') from (a') and dividing by $(x-y)$, as in the other cases, we shall obtain

$$\frac{P^{-s}-Q^{-s}}{P^{t}-Q^{t}}=A+B\left(\frac{x^{2}-y^{2}}{x-y}\right)+C\left(\frac{x^{3}-x^{3}}{x-y}\right)+,\ \&c.$$

The first member of this equation is

$$\frac{\frac{1}{P^{s}}-\frac{1}{Q^{s}}}{P^{t}-Q^{t}}=\frac{Q^{s}-P^{s}}{P^{s}Q^{s}(P^{t}-Q^{t})}=-\frac{(P^{s}-Q^{s})}{P^{s}Q^{s}(P^{t}-Q^{t})}.$$

$$\frac{-\left(\frac{P^{s}-Q^{s}}{P-Q}\right)}{P^{s}Q^{s}\left(\frac{P^{t}-Q^{t}}{P-Q}\right)}=\frac{-s\ P^{s-1}}{P^{s}Q^{s}(tP^{t-1})}=-\frac{s\ P^{s-t}}{tP^{2s}}.$$

But $$-\frac{sP^{\overline{s-t}}}{tP^{2s}}=-\frac{s}{t}P^{\overline{-s-t}}.$$

When the division is performed, and P supposed equal to Q.

Whence, $$-\frac{s}{t}P^{\overline{-s-t}}=A+2Bx+3Cx^2+, \text{\&c.}$$

But, $$P^t=1+x.$$

Therefore, $$-\frac{s}{t}P^{\overline{-s}}=A+\left.\begin{matrix}2B\\A\end{matrix}\right|x+\left.\begin{matrix}3C\\2B\end{matrix}\right|x^2+, \text{\&c.}$$

But, $$-\frac{s}{t}P^{\overline{-s}}=-\frac{s}{t}-\frac{s}{t}Ax-\frac{s}{t}Bx^2-, \text{\&c.}$$

Whence, $$A=-\frac{s}{t}\quad B=-\frac{s}{t}.\frac{-\frac{s}{t}-1}{2}, \text{\&c.}$$

and thus the demonstration is shown to be complete, under all possible circumstances.

We may use either of the three following forms :*

$$(1+x)^m=1+mx+m\frac{m-1}{2}x^2+m\frac{m-1}{2}\,\frac{m-2}{3}x^3+, \text{\&c.} \quad (1)$$

$$\left(1+\frac{x}{a}\right)^m=1+m.\frac{x}{a}+m.\frac{m-1}{2}.\frac{x^2}{a^2}+m.\frac{m-1}{2}\,\frac{m-2}{3}\,\frac{x^3}{a^3}+, \text{\&c.} \quad (2)$$

$$a^m\left(1+\frac{x}{a}\right)^m=a^m\left(1+m\frac{x}{a}+m.\frac{m-1}{2}.\frac{x^2}{a^2}+m\frac{m-1}{2}\,\frac{m-2}{3}.\frac{x^3}{a}+\right) \text{\&c.} \quad (3)$$

$$=a^m+ma^{m-1}x+m.\frac{m-1}{2}a^{m-2}x^2+m\frac{m-1}{2}\,\frac{m-2}{3}a^{m-3}x^3+, \text{\&c.}$$

APPLICATION.

1. Required the square root of $(1+x)$.

Apply formula (1), making $m=\frac{1}{2}$; then the development will be $$(1+x)^{\frac{1}{2}}=1+\tfrac{1}{2}x-\frac{x^2}{2.4}+\frac{3x^3}{2.4.6}-\frac{3.5x^4}{2.4.6.8}$$

The law of the series, in almost every case, will become apparent, after expanding three or four terms, provided we keep the factors separate.

---

* In practical examples we may be required to expand $(1-x)$ as well as $(1+x)$, and the exponent may be negative as well as positive. But in all cases the products in the second member must conform to the rules of multiplication.

The above will be the coefficients for *any binomial* square root (Art. 133); hence the square root of 2 is actually expressed in the preceding series, if we make $x=1$.

Then $(1+1)^{\frac{1}{2}}=1+\frac{1}{2}-\frac{1}{2.4}$, &c. The square root of 3 is expressed by the same series, when we make $x=2$, &c.

2. Required the cube root of $(a+b)$ or its equal, $a\left(1+\frac{b}{a}\right)$

Formula (2) gives

$$a^{\frac{1}{3}}\left(1+\frac{b}{a}\right)^{\frac{1}{3}}=a^{\frac{1}{3}}\left(1+\frac{b}{3a}-\frac{2b^2}{3.6a^2}+\frac{2.5b^3}{3.6.9a^3}-\frac{2.5.8b^4}{3.6.9.12a^4}, \text{ \&c.}\right)$$

This expresses the cube root of any binomial quantity, or any quantity that we can put into a binomial form, by giving the proper values to a and b. For example, required the cube root of 10, or its equal, $8+2$. Here $a=8$, $b=2$. Then $a^{\frac{1}{3}}=2$, and $\frac{b}{a}=\frac{1}{4}$.

Therefore, $2\left(1+\frac{1}{3.4}-\frac{2}{3.6.4^2}+\frac{2.5}{3.6.9.4^3}, \text{ \&c.}\right)$ is the cube root of 10, and so for any other number.

3. Expand $\frac{1}{a-b}$ into a series, or its equal, $(a-b)^{-1}$.

Ans. $\frac{1}{a}+\frac{b}{a^2}+\frac{b^2}{a^3}+\frac{b^3}{a^4}$, &c.

4. Expand $(a^2+b^2)^{\frac{1}{2}}$ into a series, or its equal, $a\left(1+\frac{b^2}{a^2}\right)^{\frac{1}{2}}$.

Ans. $a+\frac{b^2}{2a}-\frac{b^4}{8a^3}+\frac{b^6}{16a^5}$, &c.

5. Expand $d(c^2+x^2)^{-\frac{1}{2}}$ into a series.

Ans. $\frac{d}{c}\left(1-\frac{x^2}{2c^2}+\frac{3x^4}{2.4c^4}-\frac{3.5x^6}{2.4.6c^6}, \text{ \&c.}\right)$

6. Expand $(a^2-x^2)^{\frac{3}{4}}$ into a series.

Ans. $\sqrt{\frac{1}{a}}\left(a^2-\frac{3x^2}{2^2}-\frac{3x^4}{2^5a^2}-\frac{5x^6}{2^7a^4}, \text{\&c.}\right)$

7. Expand $(a+y)^{-4}$

Ans. $\frac{1}{a^4}-\frac{4y}{a^5}+\frac{10y^2}{a^6}-\frac{20y^3}{a^7}+\frac{35y^4}{a^8}$, &c.

8. Find the cube root of $\frac{a^3}{a^3+b^3}$.

Ans. $1-\frac{b^3}{3a^3}+\frac{2b^6}{9a^6}-\frac{14b^9}{81a^9}$, &c

9. Find the cube root of 31.

$31=27+4=27\left(1+\frac{4}{27}\right)$. *Ans.* $3\left(1+\frac{4}{27}\right)^{\frac{1}{3}}$ Place $\frac{4}{27}=a$

$(1+a)^{\frac{1}{3}}=1+\frac{1}{3}a-\frac{1}{3}\cdot\frac{2}{6}a^2+\frac{1}{3}\cdot\frac{2}{6}\cdot\frac{5}{9}a^3+$, &c.

Ans. $3\left(1+\frac{1}{3}.\frac{4}{27}-\frac{1}{3}.\frac{2}{6}.\frac{16}{(27)^2}+\frac{1}{3}.\frac{2}{6}.\frac{5}{9}.\frac{64}{(27)^3}+, \text{\&c.}\right)=3.156066$

10. Find the seventh root of 2245, or any other number, by the binomial theorem.

N. B. Find by trial the greatest seventh power, less than the given number, (2245), and divide the given number by it.

Thus, $3^7=2187$ $\quad \frac{2245}{2187}=1+\frac{58}{2187}.(2245)^{\frac{1}{7}}=3\left(1+\frac{58}{2187}\right)^{\frac{1}{7}}$

Ans. $3\left(1+\frac{1}{7}.\frac{58}{2187}-\frac{6}{7.14}.\left(\frac{58}{2187}\right)^2+\text{\&c.}\right)=3.0113574.$

11. Find the cube root of 9, by the binomial series.

$9=8(1+\frac{1}{8})$. Cube root, $2(1+\frac{1}{8})^{\frac{1}{3}}$.

Ans. 2.080084.

12. Find the cube root of 7, by the binomial series.

$7=(8-1)=8(1-\frac{1}{8})$. Cube root, $2(1-\frac{1}{8})^{\frac{1}{3}}$

Ans. 1.912933.

13. Expand $(x-y)^{-1}$, into a series.

Ans. $x^n-nx^{n-1}y+n.\frac{n-1}{2}x^{n-2}y^2-$, &c

When $n=-1$, this series becomes

$$\frac{1}{x}+\frac{y}{x^2}+\frac{y^2}{x^3}+\frac{y^3}{x^4}+, \text{ \&c., \&c.}$$

14. Find an approximate cube root of 100, by the binomial series.

N. B. The nearest cube number to 100, is 125.

Therefore, we place $\quad 100=125-25=125\left(1-\frac{25}{125}\right).$

Or, $\quad 100=125(1-\frac{1}{5})$. Whence $(100)^{\frac{1}{3}}=5(1-\frac{1}{5})^{\frac{1}{3}}$.

Ans. 4.64159, nearly.

Thus, by a little artifice, this series can be used to extract the cube (or any other root), of any number.

15. Expand $\frac{a-bx}{a+cx}$ into a series.

N. B. It is much easier to expand all such expressions as this, by actual division, than by the binomial theorem, and we take the example, merely to show that it can be expanded by the binomial theorem.

The above expression is the same as $1-\frac{(b+c)x}{a+cx}$.

Place $b+c=m$, then we are required to expand

$$mx(a+cx)^n \quad \text{or,} \quad \frac{mx}{a}\left(1+\frac{cx}{a}\right)^n$$

because $n=-1$.

But, $\left(1+\frac{cx}{a}\right)^n=1+n\frac{cx}{a}+n\frac{n-1}{2}\cdot\frac{c^2x^2}{a^2}+n\frac{n-1}{2}\cdot\frac{n-2}{3}\,\frac{c^3x^3}{a^3}+$, &c.

Because $n=-1$, this series becomes

$$1-\frac{cx}{a}+\frac{c^2x^2}{a^2}-\frac{c^3x^3}{a^3}+\frac{c^4x^4}{a^4}-\frac{c^5x^5}{a^5}\pm, \text{ \&c. \&c.}$$

this multiplied by $\frac{mx}{a}$, and the product subtracted from 1, produces $1-\frac{mx}{a}+\frac{mcx^2}{a^2}-\frac{mc^2x^3}{a^2}+\frac{mc^3x^4}{a^4}-\frac{mc^4x^5}{a^5}$, &c. the result sought, when $(b+c)$ is written in place of m.

16. Expand $\frac{x+1}{x-1}$, or its equal $1+\frac{2}{x-1}$, or its equal $1-\frac{2}{1-x}$.

Ans. $-1-2x-2x^2-2x^3-2x^4-$, &c., &c.

17. Expand $x(1-x)^{-3}$ into a series.

Ans. $x+3x^2+6x^3+10x^4+$.

18. Expand $(64+1)^{\frac{1}{3}}$, or its equal, $8(1+\frac{1}{64})^{\frac{1}{3}}$, into a series.

N. B. This is the same as demanding the cube root of 65.

CHAPTER II.

OF INFINITE SERIES.

(Art. 136.) An infinite series is a continued rank, or progression of quantities in regular order, in respect both to magnitudes and signs, and they usually arise from the division of one quantity by another.

The roots of imperfect powers, as shown by the examples in the last article, produce one class of infinite series. Some of the examples under (Art. 121.) show the *geometrical* infinite series.

Examples in common division may produce infinite series for quotients; or, in other words, we may say the division is continuous. Thus, 10 divided by 3, and carried out in decimals, gives 3.3333, &c., without end, and the sum of such a series is $3\frac{1}{3}$. (Art. 121.)

(Art. 137.) Two series may appear very different, which arise from the same source; thus 1, divided by $1+a$, gives, as we may see, by actual division, as follows:

$$
\begin{array}{l}
1+a)\,1\quad(1-a+a^2-a^3+a^4,\ \text{\&c. without end.}\\
\quad\underline{1+a}\\
\quad -a\\
\quad\underline{-a-a^2}\\
\qquad\ a^2
\end{array}
$$

Also, $a+1)1 \qquad \left(\frac{1}{a}-\frac{1}{a^2}+\frac{1}{a^3}-\frac{1}{a^4}\right.$, &c. without end.

$$\begin{array}{l} +\frac{1}{a} \\ \hline -\frac{1}{a} \end{array}$$

These two quotients appear very different, and in respect to single terms are so; but in these divisions there is always a remainder, and either quotient is incomplete without the remainder for a numerator and the divisor for a denominator, and when these are taken into consideration the two quotients will be equal.

We may clearly illustrate this by the following example:—Divide 3 by 1+2, the quotient is manifestly 1; but suppose them literal quantities, and the division would appear thus:

$$1+2)3 \quad (3-6+12, \text{\&c.}$$

$$\begin{array}{r} 3+6 \\ \hline -6 \\ -6-12 \\ \hline 12 \\ 12+24 \\ \hline -24 \end{array}$$

Again, divide the same, having the 2 stand first.

$$2+1)\ 3 \quad \left(\tfrac{3}{2}-\tfrac{3}{4}+\tfrac{3}{8}\right., \text{\&c.}$$

$$\begin{array}{r} 3+\frac{3}{2} \\ \hline -\frac{3}{2} \\ -\frac{3}{2}-\frac{3}{4} \\ \hline \frac{3}{4} \\ \frac{3}{4}+\frac{3}{8} \\ \hline -\frac{3}{8} \end{array}$$

Now let us take either quotient, with the real value of its remainder, and we shall have the same result.

Thus, $3+12=15$; and -6, and the remainder -24 divid by 3, gives —8, which make —14; hence, the whole quotient is 1

Again, $\frac{3}{2}+\frac{3}{8}=\frac{15}{8}$, and $-\frac{3}{4}-\frac{1}{8}=\frac{7}{8}$.

Hence, $\frac{15}{8}-\frac{7}{8}=\frac{8}{8}=1$, the proper quotient.

If we more closely examine the terms of these quotients, we shall discover that one is *diverging*, the other *converging*, and by the same ratio 2, and in general this is all a series can show, *the degree of convergency.*

(Art. 138.) We convert quantities into series by extracting the roots of imperfect powers, as by the binomial, and by actual division, thus:

1. Convert $\frac{a}{a+x}$ into an infinite series.

Thus, $a+x)a \quad (1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}+$, &c.

$$\begin{array}{l} a+x \\ \hline -x \\ -x-\frac{x^2}{a} \\ \hline \quad\ \ \frac{x^2}{a} \end{array}$$

2. Convert $\frac{a}{a-x}$ into an infinite series.

Ans. $1+\frac{x}{a}+\frac{x^2}{a^2}+\frac{x^3}{a^3}$, &c.

Observe that these two examples are the same, except the signs of x: when that sign is plus the signs in the series will be alternately plus and minus; when minus, all will be plus.

3. What series will arise from $\frac{1+x}{1-x}$?

Ans. $1+2x+2x^2+2x^3$ &c

Observe that in this case the series commences with $2x$. The unit is a proper quotient, and the series arises alone from $\frac{2x}{1-x}$, the remainder after the quotient 1 is obtained.

4. What series will arise from $\frac{a+x}{a^2+x^2}$?

$$a^2+x^2)\ a+x\left(\frac{1}{a}-\frac{x^2}{a^3}+\frac{x^4}{a^5}-\frac{x^6}{a^7},\ \&c.\right.$$

$$\begin{array}{l} a+\frac{x^2}{a} \\ \hline -\frac{x^2}{a}+x \\ -\frac{x^2}{a}-\frac{x^4}{a^3} \end{array}$$

Observe, in this example, the term x, in the numerator does not find a place in the operation; it will be always in the remainder; therefore, $\frac{a}{a^2+x^2}$ will give the same series.

5. What series will arise from dividing 1 by $1-a+a^2$, or from $\frac{1}{1-a+a^2}$? *Ans.* $1+a-a^3-a^4+a^6+a^7-a^9-a^{10}$, &c.

In this example, observe that the signs are not alternately plus and minus, but two terms in succession plus, then two minus; this arises from there being two terms in place of one after the minus sign in the divisor.

6. What series will arise from $\frac{a}{1-r}$?

Ans. $a+ar+ar^2+ar^3+ar^4$, &c.

Observe that this is the regular geometrical series, as appears in (Art. 118.)

7. What series will arise from $\frac{1}{1-1}$?

Ans. $1+1+1+1$, &c

That is $\frac{1}{0}$ is 1 repeated an infinite number of times, or infinity, a result corresponding to observations under (Art. 60.)

8. What series will arise from the fraction $\frac{h}{a-b}$?

Ans. $\frac{h}{a}+\frac{bh}{a^2}+\frac{b^2h}{a^3}+\frac{b^3h}{a^4}$, &c.

If $a=b$, this series will be $\frac{h}{a}$, repeated an infinite number of times.

This series can also be expanded by the binomial theorem, for $\frac{h}{a-b}=h(a-b)^{-1}$. This observation is applicable to several other examples.

(Art. 139.) A fraction of a complex nature, or having compound terms, such as $\frac{1-x}{1-2x-3x^2}$, may give rise to an infinite series, but there will be no obvious ratio between the terms. Some general relation, however, will exist between any one term and *several* preceding terms, which is called the *scale of relation*, and such a series is called a recurring series. Thus the preceding fraction, by actual division, gives $1+x+5x^2+13x^3+41x^4+121x^5$, &c., a recurring series, which, when carried to infinity, will be equal to the fraction from which it is derived.

Expand $\frac{1+2x}{1-x-x^2}$ into a series

Ans. $1+3x+4x^2+7x^3+11x^4+18x^5$, &c

CHAPTER III.

SUMMATION OF SERIES.

(Art. 140.) We have partially treated of this subject in geometrical progression, in (Art. 121); the investigation is now more general and comprehensive, and the object in some respects different. There we required the actual sum of a given number of terms, or the sum of a converging infinite series. Here the series may not be in the strictest sense geometrical, and we may not require the sum of the series, but what terms or fractional quantities will produce a series of a given *convergency*.

The object then, is the converse of the last chapter; and for every geometrical series, our rule will be drawn from the sixth example in that chapter; that is, $\frac{a}{1-r}$, a being the first term of any series, and r the ratio. *We find the ratio by dividing any term by its preceding term.*

Hence, to find what fraction may have produced any geometrical series, we have the following rule:

RULE. *Divide the first term of the series by the algebraic difference between unity and the ratio.*

EXAMPLES.

1. What fraction will produce the series 2, 4, 8, 16, &c?

Here $a=2$, and $r=2$; therefore, $\frac{2}{1-2}$ *Ans*

2. What fraction will produce the series $3-9+27-81$, &c.? Here $a=3$, and $r=-3$; then $-r=3$,

Hence, $\frac{a}{1-r}=\frac{3}{1+3}$ *Ans.*

3. What fraction will produce the series $\frac{3}{10}$, $\frac{3}{100}$, &c ?

Here $a=\frac{3}{10}$, and $r=\frac{1}{10}$; therefore,

$$\frac{\frac{3}{10}}{1-\frac{1}{10}}\times\frac{10}{10}=\frac{3}{10-1}=\frac{3}{9}=\frac{1}{3},\ \textit{Ans.}$$

4. What fraction will produce the series,

$1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}$, &c.? [See example 1, (Art. 138.)]

Here $a=1$, and $r=-\frac{x}{a}$; then $\frac{1}{1+\frac{x}{a}}=\frac{a}{a+x}$, *Ans*

5. What fraction will produce the series. $1+2x+2x^2+2x^3$, &c.? [See example 3, (Art. 138.) and the observation in connection.]

Here $a=2x$, $r=x$; then $\frac{2x}{1-x}$, will give all after the first term: therefore, $1+\frac{2x}{1-x}=\frac{1+x}{1-x}$, *Ans.*

6. What fraction will produce the series $\frac{d}{c}-\frac{db}{c^2}+\frac{db^2}{c^3}-\frac{db^3}{c^4}$, &c.?

Ans. $\frac{d}{c+b}$.

7. What fraction will produce the series $\frac{d}{b}+\frac{adx}{b^2}+\frac{a^2dx^2}{b^3}$, &c.

Ans. $\frac{d}{b-ax}$.

8. What fraction will produce the series

$2b-\frac{2b^2}{a}+\frac{2b^3}{a^2}-\frac{2b^4}{a^3}+$, &c *Ans.* $\frac{2ab}{a+b}$.

9. What fraction will produce the series

$1+a-a^3-a^4+a^6+a^7-a^9-a^{10}$, &c.?

See example 5, (Art. 138.)

Put $1+a=b$; then $a^3+a^4=a^3b$, and $a^6+a^7=a^6b$, and the series becomes $b-a^3b+a^6b-$, &c.

Hence, the fraction required is $\frac{b}{1+a^3}=\frac{1+a}{1+a^3}=\frac{1}{1-a+a^2}$.

10. What fraction will produce the series $x+x^2+x^3$, &c.?

Ans. $\frac{x}{1-x}$.

If $x=1$, this expression becomes $\frac{1}{1-1}=\frac{1}{0}$, a symbol of infinity

11. What fraction will produce the series $1+\frac{3}{5}+\frac{9}{25}$, &c.?

Ans. $\frac{1}{1-\frac{3}{5}}=\frac{5}{5-3}$.

Hence, the sum of this series, carried to infinity, is $2\frac{1}{2}$.

In the same manner, we may resolve every question in (Art. 121.)

12. What is the sum of the series, or what fraction will produce the series $1-a+a^3-a^4+a^6-a^7+a^9-$, &c.?

Ans. $\frac{1}{1+a+a^2}$.

13. What is the value of the infinite series

$$\frac{6}{10}+\frac{6}{(10)^2}+\frac{6}{(10)^3}, \text{ \&c.}$$

Ans. $\frac{2}{3}$.

RECURRING SERIES.

(Art. 141.) We have explained recurring series in (Art. 139.) and it is evident that we cannot find their equivalent fractions by the operation which belongs to the geometrical order, as no common relation exists between the single terms. The fraction $\frac{1+2x}{1-x-x^2}$, by actual division, gives the series $1+3x+4x^2+7x^3+11x^4+18x^5$, &c., without termination; or, in other words, the division would continue to infinity.

Now, having a few of these terms, it is desirable to find a method of deducing the fraction.

There is no such thing as deducing the fraction, or in fact no fraction could exist corresponding to the given series, unless *order* or a law of dependence exists among the terms; therefore some order must exist, but that order is not apparent.

Let the given series be represented by

$$A+B+C+D+E+F, \text{ \&c.}$$

Two or three of these terms must be given, and then each succeeding term may depend on two or three or more of its preceding terms.

In cases where the terms depend on *two* preceding terms we may have

$$\left.\begin{array}{r} C=mBx+nAx^2 \\ D=mCx+nBx^2 \\ E=mDx+nCx^2 \\ \text{\&c.}=\text{\&c} \end{array}\right\} \qquad (1)$$

In cases where the terms, or law of progression depend on *three* preceding terms we may have

$$\left.\begin{array}{l} D=mCx+nBx^2+rAx^3 \\ E=mDx+nCx^2+rBx^3 \\ F=mEx+nDx^2+rCx^3 \\ \&c.=\&c. \end{array}\right\} \quad (2)$$

The reason of the regular powers of x coming in as factors, will be perfectly obvious, by inspecting any series.

The values of m, n and r express the unknown relation, or law that governs the progression, and are called the scale of relation We shall show how to obtain the values of these quantities in a subsequent article.

(Art. 142.) Let us suppose the series of equations (1), to be extended indefinitely, or, as we may express it, to infinity, and add them together, representing the entire sum of $A+B+C+D$, &c., to infinity, by S; then the first member of the resulting equation must be $(S-A-B)$, and the other member is equally obvious, giving

$$S-A-B=mx(S-A)+nx^2S$$

$$\text{Hence,} \quad S=\frac{A+B-mAx}{1-mx-nx^2} \qquad (a)$$

In the same manner, from equations (2), we may find

$$S=\frac{A+B+C-(A+B)mx-Anx^2}{1-mx-nx^2-rx^3} \qquad (b)$$

(Art. 143.) The *form* of a series does not depend on the value of x, and any series is true for all values of x. Equations (1) then, will be true, if we make $x=1$.

Making this supposition, and taking the first two equations of the series (1), we have

$$\left.\begin{array}{l} C=mB+nA \\ \text{And} \quad D=mC+nB \end{array}\right\} \qquad (c)$$

In these equations, A, B, C, D, are known, and m and n unknown; but two unknown quantities can be determined from two equations; hence m and n can be determined

When the scale of relation depends upon three terms, we take three of the equations (2), making $x=1$, and we determine m, n, and r, as in simple equations.

EXAMPLES.

1. What fraction will produce the series

$$1+3x+4x^2+7x^3+11x^4, \text{ \&c.?}$$

To determine the scale of relation, we have $A=1$, $B=3$, $C=4$, and $D=7$. Then from equations (c), we have

$$n+3m=4$$
$$3n+4m=7$$

These equations give $m=1$, and $n=1$.

Now to apply the general equation (a), we have $A=1$, $B=3x$.

Then $S=\dfrac{A+B-mAx}{1-mx-nx^2}=\dfrac{1+3x-x}{1-x-x^2}=\dfrac{1+2x}{1-x-x^2}$, *Ans.*

2. What fractional quantity will produce the series

$$1+6x+12x^2+48x^3+120x^4, \text{ \&c., to infinity?}$$

Here $A=1$, $B=6x$, $m=1$, $n=6$.

Hence, by applying equation (a), we find $\dfrac{1+5x}{1-x-6x^2}$, for the expression required.

3. What quantity will produce the series

$1+3x+5x^2+7x^3+9x^4$, to infinity? *Ans.* $\dfrac{1+x}{(1-x)^2}$

4. What quantity will produce the infinite series $1+4x+6x^2+11x^3+28x^4+63x^5$, &c.? in which $m=2$, $n=-1$, $r=3$.

[Apply equation (b).] *Ans.* $\dfrac{(1+x)^2-2x^3}{(1-x)^2-3x^3}$.

5. What fraction will produce the series

$$1+x+5x^2+13x^3+41x^4+121x^5, \text{ \&c.?}$$

Ans. $\dfrac{1-x}{1-2x-3x^2}$

6. What fraction will produce the series

$$1+3x+2x^2-x^3-3x^4-2x^5+x^6,\ \&c.\,?$$

Ans. $\dfrac{1+2x}{1-x+x^2}$.

REVERSION OF A SERIES.

(Art. B.) To revert a series is to express the value of the unknown quantity in it, (which appears in the several terms under regular powers,) by means of another series involving the powers of some other quantity.

Thus, let x and y represent two indeterminate quantities, and the value of y be expressed by a series involving the regular powers of x. That is

$$y=ax+bx^2+cx^3+dx^4\ \&c.$$

To revert this, is to obtain the simple value of x, by means of another series containing only the known quantities, a, b, c, d, &c. and the powers of y.

To accomplish this, *assume*

$$x=Ay+By^2+Cy^3,\ \&c.^*$$

Substitute the assumed value of x for x, x^2, x^3, &c., in the proposed series, and transposing y, we shall have

$$0=\left\{\begin{array}{l|l|l|l|l} aA & y+aB & y^2+aC & y^3+aD & y^4+,\ \&c.\\ -1 & \quad+bA^2 & \quad+2bAB & \quad+2bAC & \\ & & \quad+cA^3 & \quad+bB^2 & \\ & & & \quad+3cA^2B & \\ & & & \quad+dA^4 & \end{array}\right.$$

---

* By examining previous authors, we have found none that explain the *rationale* of such assumptions: but these are points on which the learner requires the greatest profusion of light. We explain thus: the term x must have some value, *positive*, *negative*, or *zero*, and the series $Ay+By^2+Cy^3$, &c., can be positive and of any magnitude. It can be negative, and of any magnitude, by giving the coefficients A, B and C, negative values. It can be zero by making $y=0$. Therefore it can express the value of x, whatever that may be; and because the powers of y are regular, the substitution of this value of x for the several powers of x, in the primitive series, will convert that series into regular powers of y, which was the object to be accomplished.

As every term contains y, we can reduce the equation by dividing by y; and afterwards, in consideration that the equation must be true for all values of y: making $y=0$, we shall have

$$aA-1=0 \qquad \text{or} \qquad A=\frac{1}{a}$$

Continuing the same operation and mode of reasoning as in (Art. 128.), and we shall find, in succession, that

$$(F)\begin{cases} A=\dfrac{1}{a} \\ B=-\dfrac{b}{a^3} \\ C=\dfrac{2b^2-ac}{a^5} \\ D=-\dfrac{5b^3-5abc+a^2d}{a^7} \\ \&c. = \&c. \end{cases}$$

Substituting these values of A, B, C, &c., in the assumed equation and we have the value of x in terms of a, b, c, &c., and the powers of y as required, or a complete reversion of the series.

EXAMPLES.

1. Revert the series $y=x+x^2+x^3$, &c.

Here $a=1$ $b=1$ $c=1$, &c.

Assume $x=Ay+By^2+Cy^3+$&c.

Result, $x=y-y^2+y^3-y^4+y^5-$&c.

2. Revert the series $x=y-\dfrac{y^2}{2}+\dfrac{y^3}{3}-\dfrac{y^4}{4}$ &c.

Here $a=1$ $b=-\frac{1}{2}$ $c=\frac{1}{3}$ $d=-\frac{1}{4}$ &c.

Assume $y=Ax+Bx^2+Cx^3+Dx^4+$ &c., and find the values of A, B, C, &c., from the formulas (F).

Result, $y=x+\dfrac{x^2}{1.2}+\dfrac{x^3}{1.2.3}+\dfrac{x^4}{1.2.3.4}$. &c

3. Revert the series $y=x+3x^2+5x^3+7x^4+9x^5$, &c.

Result $x=y-3y^2+13y^3-67y^4$.

In case the given series is of the form of $x=ay+by^3+cy^5$, &c., the powers of y varying by 2, the equations (F) will not apply, and we must assume $y=Ax+Bx^3+Cx^5$, &c., and substitute as before, and we shall find

$$(G)\begin{cases} A=\dfrac{1}{a} \\ B=-\dfrac{b}{a^4} \\ C=\dfrac{3b^2-ac}{a^7} \\ D=-\dfrac{(12b^3+a^2d-8abc)}{a^{10}} \end{cases}$$

In common cases, after the coefficients, as far as D, are determined, the law of continuation will become apparent, especially if the factors are kept separate.

CHAPTER IV

EXPONENTIAL EQUATIONS AND LOGARITHMS.

(Art. 144.) We have thus far used exponents only as known quantities; but an exponent, as well as any other quantity, may be variable and unknown, and we may have an equation in the form of $a^x=b$.

This is called an exponential equation, and the value of x can only be determined by successive approximations, or by making use of a table of logarithms already determined.

(Art. 145.) *Logarithms are exponents.* A given constant number may be conceived to be raised to all possible powers, and thus produce all possible numbers; the exponents of such powers are logarithms, each corresponding to the number produced.

Thus, in the equation $a^x=b$, x is the logarithm of the number b; and to every variation of x, there will be a corresponding variation to b; a is constant, and is called the base of the system, and differs only in different systems.

The constant a cannot be 1, for every power of 1 is 1, and the variation of x in that case would give no variation to b; hence, the base of a system cannot be unity; in the common system it is 10.

In the equation $10^x=2$, x is, in value, a small fraction, and is the logarithm of the abstract number 2.

In the equation $a^x=b$, if we suppose $x=1$, the equation becomes $a^1=a$; that is, the logarithm of the base of any system is unity.

If we suppose $x=0$, the equation becomes $a^0=1$; hence, the logarithm of 1 is 0, in every system of logarithms.

(Art. 146.) *The logarithms of two or more numbers added together give the logarithm of the product of those numbers, and conversely the difference of two logarithms gives the logarithm of the quotient of one number divided by the other.*

For we may have the equations $a^x=b$, $a^y=b'$, and $a^z=b''$.

Multiply these equations together, and as we multiply powers by *adding the exponents* the product will be

$$a^{x+y+z}=bb'b''$$

Hence, by the definition of logarithms, $x+y+z$ is the logarithm for the number represented by the product $bb'b''$. Again. divide the first equation by the second, and we have $a^{x-y}=\frac{b}{b'}$; and from these results we find that by means of a table of logarithms *multiplication may be practically performed by addition, and division by subtraction*, and in this consists the great utility of logarithms.

(Art. 147.) In the equation $a^x=b$, take $a=10$, and x successively equal to 0, 1, 2, 3, 4, &c.

Then $10^0=1$, $10^1=10$, $10^2=100$, $10^3=1000$, &c.

Therefore, for the numbers

1, 10, 100, 1000, 10000, 100000, &c., we have for corresponding logarithms

0, 1, 2, 3, 4, 5, &c.

Here it may be observed that the numbers increase in geometrical progression, and their logarithms in arithmetical progression.

Hence the number which is the geometrical mean between two given numbers must have the arithmetical mean of their logarithms, for its logarithm.

On this principle we may approximate to the logarithm of any proposed number. *For example, we propose to find the logarithm of* 2.

This number is between 1 and 10, and the geometrical mean between these two numbers, (Art. 122.), is 3.16227766. The arithmetical mean between 0 and 1 is 0.5; therefore, the number 3.16227766 has 0.5 for its logarithm.

Now the proposed number 2 is between 1 and 3.162, &c., and the geometrical mean between these two numbers is 1.778279, and the arithmetical mean between 0. and 0.5 is 0.25; therefore, the logarithm of 1.778279 is 0.25.

Now the proposed number 2 lies between 1.778279, and 3.16227766, and the geometrical mean between them will fall near 2, a little over, and its logarithm will be 0.375. Continuing the approximations, we shall at length find the logarithm of 2 to be 0.301030, and in the same manner we may approximate to the logarithm of any other number, but the operation would be very tedious.

(Art. 148.) We may take a reverse operation, and propose a logarithm to find its corresponding number; thus, in the general equation $a^x=n$; x may be assumed, and the corresponding value of n computed.

Thus suppose $x=\frac{3}{10}$; then $(10)^{\frac{3}{10}}=n$, or $10^3=n^{10}$.

$$\text{Hence, } n=\sqrt[10]{1000}=1.995262315.$$

That is, the number 1.9952, &c., (nearly 2) has 0.3 for its logarithm. In the same way we may compute the numbers corresponding to the logarithms 0.4, 0.5, 0.6, 0.7, &c.

(Art. 149.) We may take another method of operation to find the logarithm of a number.

Let the logarithm of 3 be required.

The equation is $10^x=3$, the object is to find x.

It is obvious that x must be a fraction, for $10^1=10$; therefore,

Let $x=\frac{1}{x'}$. Then $10^{\frac{1}{x'}}=3$, or $10=3^{x'}$. Here, we perceive, that x' must be a little more than 2. Make $x'=2+\frac{1}{x''}$. Then $3^2\times 3^{\frac{1}{x''}}=10$; or $3^{\frac{1}{x''}}=\frac{10}{9}$.

$$\text{Or } \left(\frac{10}{9}\right)^{x''}=3.$$

Here we find by trial that x'' is between 10 and 11; take it 10, and $x'=2+\frac{1}{10}$; hence, $x=\frac{10}{21}=0.476$, for a rough approximation for the logarithm of 3. A more exact computation gives 0.4771213; but all these operations are exceedingly tedious, and to avoid them, mathematicians have devised a more expeditious method by means of a converging series; which we shall investigate in a subsequent article.

(Art. 150.) It is only necessary to calculate directly the logarithms of prime numbers, as the logarithms of all others may be derived from these. Thus, if we would find the logarithm of 4, we have only to double that of 2; for, taking the equation $(10)^x=2$, and squaring both members we have, $(10)^{2x}=4$; or taking the same equation, and cubing both members, we have $(10)^{3x}=8$; which shows that twice the logarithm of 2 is the logarithm of 4, and three times the logarithm of 2 is the logarithm of 8; and, in short, the sum of two or more logarithms corresponds to the logarithm of the product of their numbers, (Art. 142.), and the difference of two logarithms corresponds to the logarithm of the quotient, which is produced from dividing one number by the other.

Thus, the logarithm of $\frac{a}{b}=\log. a-\log. b$. The abbreviation, ($\log. a$), is the symbol for the logarithm of a.

(Art. 151.) As the logarithm of 1 is 0, 10 is 1, 100 is 2, &c., we may observe that the whole number in the logarithm is *one* less than the number of places in the number.

The whole number in a logarithm is called its *characteristic*, and is not given in the tables, as it is easily supplied. For example, the integral part of the logarithm of the number 67430 must be 4, as the number has 5 places. The same figures will have the same decimal part for the logarithm when a portion of them become decimal.

Thus,	67430	logarithm	4.82885
	6743.0		3.82885
	674.30		2.82885
	67.430		1.82885
	6.7430		0.82885
	.67430		—1.82885

For every division by 10 of the number, we must diminish the characteristic of the logarithm by unity.

The decimal part of a logarithm is always positive; the index or characteristic becomes *negative*, when the number becomes less than unity.

By reference to (Art. 18.), we find that $\frac{1}{10}=10^{-1}$, $\frac{1}{100}=\frac{1}{10^2}=10^{-2}$, &c. That is: fractions may be considered as numbers with *negative* exponents, and logarithms are exponents; therefore the logarithm of $\frac{1}{10}$, or .1, is —1; of $\frac{1}{100}$, or .01, is —2, &c. If any addition is made to .01 the logarithm must be more than —2; but, for convenience, we still let the index remain —2, and make the decimal part *plus*. *Hence the index alone must be considered as minus.*

Negative numbers have no logarithms; for, in fact, as we have before observed, there are no such numbers.

(Art. C.) *We now design to show a practical method of computing logarithms.* The methods hitherto touched upon were only designed to be explanatory of the nature of logarithms; but, to calculate a table, by either of those methods, would exhaust the patience of the most indefatigable. To arrive at easy practical results requires the clearest theoretical knowledge, and we must therefore frequently call the attention of students to *first principles.*

The fundamental equation is $a^x=b$, in which a is the constant base and x is the logarithm of the number b; and b may be of any magnitude or in any form, if it be essentially *positive.*

Now we may take $a=1+c$ and $b=1+\frac{1}{p}$

Then the fundamental equation becomes $(1+c)^x=\left(1+\frac{1}{p}\right)$ (1)

Here $x=\log.\left(1+\frac{1}{p}\right)=\log.\left(\frac{p+1}{p}\right)=\log.(p+1)-\log.p$ (2)

Raise both members of equation (1) to the nth power, then we shall have

$$(1+c)^{nx}=\left(1+\frac{1}{p}\right)^n$$

Expand both members into a series by the *Binomial Theorem,*

$$\text{Then}\quad 1+nxc+nx.\frac{nx-1}{2}c^2+nx.\frac{nx-1}{2}.\frac{nx-2}{3}c^3+$$

$$nx.\frac{nx-1}{2}.\frac{nx-2}{3}.\frac{nx-3}{4}c^4+,\ \&c.\ =\ 1+n.\frac{1}{p}+n.\frac{n-1}{2}.\frac{1}{p^2}+$$

$$n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{1}{p^3}+n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}.\frac{1}{p^4}+,\ \&c.$$

Drop unity, from both members, and divide by n, then we have

$$x\left(c+\frac{nx-1}{2}c^2+\frac{nx-1}{2}.\frac{nx-2}{3}c^3+\frac{nx-1}{2}.\frac{nx-2}{3}.\frac{nx-3}{4}c^4\right.$$

$$\left.+\&c.\right)\ =\ \frac{1}{p}+\frac{n-1}{2}.\frac{1}{p^2}+\frac{n-1}{2}.\frac{n-2}{3}.\frac{1}{p^3}+\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}.\frac{1}{p^4}$$

$+$, &c.

This equation is true for all values of n. It is, therefore, true when $n=0$. Making this supposition and the above equation reduces to

$$x\left(c-\frac{c^2}{2}+\frac{c^3}{3}-\frac{c^4}{4}+\frac{c^5}{5}-\&c.\right)=\frac{1}{p}-\frac{1}{2p^2}+\frac{1}{3p^3}-\frac{1}{4p^4}+\frac{1}{5p^5}-\&c. \quad (A)$$

As c remains a constant quantity for all variations of x and p, the series in the vinculum may be represented by the symbol M

$$\text{Then } xM=\frac{1}{p}-\frac{1}{2p^2}+\frac{1}{3p^3}-\frac{1}{4p^4}+\frac{1}{5p^5}-, \&c. \qquad (3)$$

Take the value of x from equation (2), and substitute it in equation (3),

$$\text{Then } [\log.(p+1)-\log.p]M=\frac{1}{p}-\frac{1}{2p^2}+\frac{1}{3p^3}-\frac{1}{4p^4}+, \&c. \quad (4)$$

Again, we may have the fundamental equation $(1+c)^y=\left(1-\frac{1}{p}\right)$, in which y is the logarithm of $\left(1-\frac{1}{p}\right)$, the same as x was of $\left(1+\frac{1}{p}\right)$.

$$\text{Or } \quad y=\log.\left(\frac{p-1}{p}\right)=\log.(p-1)-\log.\,p. \qquad (5)$$

Operating on the equation $(1+c)^y=1-\frac{1}{p}$, the same as we did on equation (1), we shall find

$$[\log.(p-1)-\log.\,p]M=-\frac{1}{p}-\frac{1}{2p^2}-\frac{1}{3p^3}-\frac{1}{4p^4}-, \&c. \qquad (6)$$

Subtract equation (6) from equation (4), and we obtain

$$[\log.(p+1)-\log.(p-1)]M=2\left(\frac{1}{p}+\frac{1}{3p^3}+\frac{1}{5p^5}+\frac{1}{7p^7}, \&c.\right) \quad (7)$$

Dividing by M, and considering that $\log.\left(\frac{p+1}{p-1}\right)=\log.(p+1)-\log.(p-1)$, the equation can take this form

$$\log.\left(\frac{p+1}{p-1}\right)=\frac{2}{M}\left(\frac{1}{p}+\frac{1}{3p^3}+\frac{1}{5p^5}+\frac{1}{7p^7}+\frac{1}{9p^9}, \&c.\right) \qquad (8)$$

As p may be any positive number, greater than 1, make $\frac{p+1}{p-1}=1+\frac{1}{z}$. Then $p=2z+1$, and equation (8) becomes

$$\log.\left(\frac{z+1}{z}\right)=\log.(z+1)-\log.z=$$

$$\frac{2}{M}\left(\frac{1}{2z+1}+\frac{1}{3(2z+1)^3}+\frac{1}{5(2z+1)^5}, \&c.\right) \ldots (9)$$

By this last equation we perceive that the logarithm of $(z+1)$ will become known when the log. of z is known, and some value assigned for the constant M. Baron Napier, the first discoverer of logarithms, gave M the arbitrary value of unity, for the sake of convenience.

Then, as in every system of logarithms, the logarithm of 1 is 0, make $z=1$, in equation (9), and we shall have

$$\log.2=2\left(\frac{1}{3}+\frac{1}{3(3)^3}+\frac{1}{5(3)^5}+, \&c.\right)=.0.69314718.$$

This is called the Napierian logarithm of 2, because its magnitude depends on Napier's base, or on the particular value of M being unity.

Having now the Napierian logarithm of 2, equation (9) will give us that of 3. Double the log. of 2 will give the logarithm of 4. Then, with the log. of 4, equation (9) will give the logarithm of 5, and the log. of 5 added to the log. of 2, will give the logarithm of 10.

Thus the Napierian logarithm of 10 has been found to great exactness, and is 2.302585093.

The Napierian logarithms are not convenient for arithmetical computation, and Mr. Briggs converted them into the common logarithms, of which the base is made equal to 10.

To convert logarithms from one system into another, we may proceed as follows:

Let e represent the Napierian base, and a the base of the common system, and N any number.

Also, let x represent the logarithm of N, corresponding to the base a, and y the logarithm of N, corresponding to the base e.

$$\text{Then} \quad a^x=N,$$
$$\text{and} \quad e^y=N.$$

Now, by inspecting these equations, it is apparent that if the base a is greater than the base e, the log. x will be less than the log. y.

These equations give $a^x=e^y$.

Taking the logarithms of both members, observing that x and y are logarithms already, we have

$$x \log. a=y \log. e.$$

This equation is true, whether we consider the logarithms taken on the one base or on the other. Conceive them taken on the common base, then

$$\log. a=1, \quad \text{and} \quad x=y \log. e \ldots\ldots (10)$$

$$\text{or} \quad \log. e=\frac{x}{y}.$$

In this equation x and y must be logarithms of the same number, and therefore if we take $x=1$, which is the logarithm of 10, in the common system, y must be 2.302585093, as previously determined.

$$\text{Hence } \log. e=\frac{1}{2.302585093}=0.434294482 \ldots \quad (11)$$

This last decimal is called the *modulus* of the common system; for by equation (10) we perceive that it is the constant multiplier to convert Napierian or hyperbolic logarithms into common logarithms.

But equation (9) gives Napierian logarithms when $M=1$; therefore the same equation will give the common logarithms by causing M to disappear, and putting in this decimal as a factor.

Equation (9) becomes the following formula for computing common logarithms:

$$\log.(z+1)-\log.z=$$

$$0.86858896\left(\frac{1}{2z+1}+\frac{1}{3(2z+1)^3}+\frac{1}{5(2z+1)^5}+,\ \&c.\right)\ (F)$$

To apply this formula, assume $z=10$. Then

$$\log.\ z=1, \qquad \text{and} \qquad 2z+1=21.$$

$$\begin{array}{r|rcl}
21 & 0.86858896 & & \\
21^2=441 & 0.041361379\div 1 & = & 0.041361379 \\
441 & 93792\div 3 & = & 31264 \\
 & 212\div 5 & = & 42 \\
\hline
 & & & .041392685=\text{sum of series.} \\
 & \log.10= & & 1.0 \\
\hline
 & \log.(z+1)= & & 1.041392685=\log.11.
\end{array}$$

If we make $z=99$, then $(z+1)=100$, and $\log.(z+1)=2$, and $2z+1=199$.

$$\begin{array}{r|rl}
199 & 0.86858896 & \\
39601 & 436477\div 1= & 0.00436477 \\
 & 11\div 3= & 4 \\
\hline
 & & 0.00436481=\text{sum of series.}
\end{array}$$

$$\begin{array}{lrl}
\text{Hence} & 2.00000-\log.99= & 0.00436481 \\
\text{By transposition} & \log.99= & 1.99563519 \\
\text{Subtract} & \log.11= & 1.04139269=\log.11 \\
\hline
 & \log.\ 9= & 0.95424234 \\
 & \tfrac{1}{2}\log.9=\log.\ 3= & 0.47712117=\log.3.
\end{array}$$

Thus we may compute logarithms with great accuracy and rapidity, using the formula for the prime numbers only.

By equation (11) we perceive that the logarithm of the Napierian base is 0.434294482; and this logarithm corresponds to the number 2.7182818, which must be the base itself.

We may also determine this base directly:

In the fundamental equation (1), the base is represented by $(1+c)$. In equation (A), c must be taken of such a value as shall make the series $c-\frac{c^2}{2}+\frac{c^3}{3}-\frac{c^4}{4}+$, &c., equal to 1. But to determine what that value shall be, in the first place, put

$$y=c-\frac{c^2}{2}+\frac{c^3}{3}-\frac{c^4}{4}, \text{ \&c.}$$

Now by reverting the series (Art. B.), we find that

$$c=y+\frac{y^2}{1.2}+\frac{y^3}{1.2.3}+\frac{y^4}{1.2.3.4}, \text{ \&c.}$$

But, by hypothesis, the series involving c equals unity that is, $y=1$. Therefore

$$c=1+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}, \text{ \&c.}$$

By taking 12 terms of this series, we find $(1+c)=2.7182818$, the same as before.

TABLE I.—LOGARITHMS FROM 1 TO 100.

N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0 000000	26	1 414973	51	1 707570	76	1 880814
2	0 301030	27	1 431364	52	1 716003	77	1 886491
3	0 477121	28	1 447158	53	1 724276	78	1 892095
4	0 602060	29	1 462398	54	1 732394	79	1 897627
5	0 698970	30	1 477121	55	1 740363	80	1 903090
6	0 778151	31	1 491362	56	1 748188	81	1 908485
7	0 845098	32	1 505150	57	1 755875	82	1 913814
8	0 903090	33	1 518514	58	1 763428	83	1 919078
9	0 954243	34	1 531479	59	1 770852	84	1 924279
10	1 000000	35	1 544068	60	1 778151	85	1 929419
11	1 041393	36	1 556303	61	1 785330	86	1 934498
12	1 079181	37	1 568202	62	1 792392	87	1 939519
13	1 113943	38	1 579784	63	1 799341	88	1 944483
14	1 146128	39	1 591065	64	1 806180	89	1 949390
15	1 176091	40	1 602060	65	1 812913	90	1 954243
16	1 204120	41	1 612784	66	1 819544	91	1 959041
17	1 230449	42	1 623249	67	1 826075	92	1 963788
18	1 255273	43	1 633468	68	1 832509	93	1 968483
19	1 278754	44	1 643453	69	1 838849	94	1 973128
20	1 301030	45	1 653213	70	1 845098	95	1 977724
21	1 322219	46	1 662578	71	1 851258	96	1 982271
22	1 342423	47	1 672098	72	1 857333	97	1 986772
23	1 361728	48	1 681241	73	1 863323	98	1 991226
24	1 380211	49	1 690196	74	1 869232	99	1 995635
25	1 397940	50	1 698970	75	1 875061	100	2 000000

TABLE II.—Logarithms of Leading Numbers without Indices.

	N. 100.	N. 101.	N. 102.	N. 103.	N. 104.	N. 105.	N. 106.	N. 107.	N. 108.	N. 109.
0	000000	004321	008600	012837	017033	021189	025306	029384	033424	037426
1	000434	004750	009026	013259	017451	021603	025715	029789	033826	037825
2	000867	005881	009451	013680	017868	022016	026125	030195	034227	038223
3	001301	005609	009876	014100	018284	022428	026553	030600	034628	038620
4	001734	006038	010300	014521	018700	022841	026942	031004	035029	039017
5	002166	006466	010724	014940	019116	023252	027350	031408	035430	039414
6	002598	006894	011147	015360	019532	023664	027757	031812	035830	039811
7	003029	007321	011570	015779	019947	024075	028164	032216	036230	040207
8	003461	007748	011993	016197	020361	024486	028571	032619	036629	040602
9	003891	008174	012415	016616	020775	024896	028978	033021	037028	040998

(Art. 151*a*.) The preceding tables are sufficient to give us the logarithms of any number whatever, provided we understand the theory of logarithms, and are able to use a few practical artifices.

If to this knowledge, and these artifices, we add the formula, we can with ease find the log. of any number, however large, or however small, to any required degree of accuracy.*

We can, in fact, find the logarithm of any number, very nearly, by Table I.

ILLUSTRATIONS.

1. *We require the logarithm of* 520; but 520 is not in the table, 52 and 10 are.

Log. of 52	is	1.716003.
Log. of 10	is	1.000000.
Log. of 520, therefore, is		2.716003.

2 The logarithm of 5.2 is, 0.716003,
And of .52 is —1.716003, &c., &c.

3. *What is the logarithm of* 146?

$146 = 73 \times 2$. Whence log. 73, 1.863323
log. 2, 301030

Log of 146 is 2.164353

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*It is very bad policy to give a learner the use of voluminous tables; such tables are made to supply the place of theory, and of thought. Contracted tables, like these, are the best educators.

4. *What is the logarithm of* 3244?

3240=60·54.		3250=65·50.	
Log. 60,	1.778151	Log. 65,	1.812913
Log. 54,	1.732394	Log. 50,	1.698970
Log. 3240, is	3.510545.	Log. 3250, is	3.511883.

(Art. 151*b*.) Here we have the logarithm of 3240, and of 3250; and the number 3244 is *intermediate*, therefore its logarithm must be intermediate between 3.510545 and 3.511883. The difference between the two extreme numbers is 10, and the difference of their logarithms is .001338.

Therefore, by proportion, 10 : .001338 :: 4.

That is $\frac{4}{10}$ths of the difference of the known logarithms is the correction to add to the less logarithm to obtain the logarithm sought.

Whence to log. 3240	3.510545
Add .0001338×4	535
Log. of 3244 is	3.511080

This example shows that Table I., and the true theory of logarithms, are sufficient to find the logarithm of any number whatever.

ANOTHER METHOD.

(Art. 151*c*.) When the number is large, as in the present example, the first term of the series will be the increase of the logarithm corresponding to one; four times this would be the increase corresponding to four, and so on.

The logarithm of 3240 is 3.510545, as before found, and here $z=3240$, and $2z+1=6481$,

Whence, $\frac{.86858896}{6481}=0.000134$

Whence to log.		3.540545
Add	0.000134×4	536
Log. 3244		3.511081

And the two methods agree—the first may be clearest to a learner, but the second is more brief and scientific.

(Art. 151*d*.) Table II. requires some explanation. The numbers are at the top and side, and only the decimal part of the logarithm is contained in the table. The operator, in each particular case, will supply the index, according to the theory of logarithms.

EXAMPLES.

1. *Find the logarithm of* 1013. *Ans.* 3.005609.

We find 101 at the top, and 3 at the side. Under the former, and opposite the latter, we find the decimal .005609, and this is the decimal part of the logarithms of all numbers expressed by the figures 1013, as they here stand.

Thus, it is the decimal logarithm of 1013, 101.3, 10.13, 1.013, or of the decimals .1013, .01013, &c.

It is also the decimal logarithm of 10130, 101300, &c., but this was sufficiently illustrated in (Art. 151).

2. *Find the logarithm of* 1.075. *Ans.* 0.031408.

3. *Find the logarithm of* 10940. *Ans.* 4.039017.

Here, under 109, at the top, and opposite 4, at the side, we find the decimal, and we take 4 for the index, because there are five places of figures, in whole numbers.

4. *Find the logarithm of* 107.5. *Ans.* 2.031408.

5. *Find the logarithm of* .001043. *Ans.* $\bar{3}$.018284.

(Art. 151*e*.) Table II. will be very useful in solving problems in compound interest, and annuities; but its greatest utility is in solving the following problems:

6. *Find the logarithm of* 567521. *Ans.* 5.753977.

Divide by the two *superior figures* (in this example it is 56), and the quotient is $10134\frac{17}{56}$.

The divisor, consisting of two figures, its logarithm can always be found in Table I.

And the four *superior figures* of the quotient (any quotient) will be found in Table II.

The quotient in this example may be written thus,

$$10130+4\tfrac{17}{56}.$$

The logarithm of 10130 (the decimal part of it), can be taken directly out of Table II. It is .005609; and the next greater (the number below) is .006038; the difference between the two is .000429, and the correction for 4$\frac{17}{56}$ must be 4$\frac{17}{56}$ times one tenth of .000429, which is .000180.

Whence log. 10130 - - - -	4.005609	Table II.
Correction for 4$\frac{17}{56}$, - -	.000180	
Log. of quotient, 10134$\frac{17}{56}$, -	4.005789	
Log. of 56, - - - - - -	1.748188	Table I.
Log. of 567521,	5.753977.	

Thus we may find the logarithm of any number by the use of these two tables. We give one more example.

7. *Find the logarithm of* 365.25638—the days and parts of a day in a siderial year.

(N. B. As three figures are whole numbers, the index must be two. Divide by 36, and taking five figures in the quotient as whole numbers, we shall have 10146.01 for a quotient.)

This quotient may be written in the following form:

$$10140+6.01.$$

The Log. of 36, Table I., - -	0.556303
Log. of 10140, Table II., -	0.006038
Correction for $\frac{6.01}{10}$ of diff.*	.000257
Sum $+2.=$log. of 365.25638,	2.562598

(Art. 151*f.*) To find the number corresponding to a given logarithm we take the converse operation.

For example, we take the logarithm just found, and demand its corresponding number.

Log. (omit index), - -		0.562598
Next less in Table I.	36, -	.556303
		.006295
Next less in Table II.	10140,	.006038
		.000257

---

* The difference in the table is the difference between .006038 and the next greater .006466, which is .000428, one-tenth of this .000042.8, multiply by (6.01) and the product is .000257.

Divide this difference by the tabular difference, which is .000428, and we obtain 6, and a very small remainder.

Whence, to 10140, add 6, and we have 10146, which being multiplied by 36, produces 365.256, for the number sought.

We take three of the figures for whole numbers, because the index of the given logarithm is 2. Had it been 3, we should have taken four figures for whole numbers.

1. *What number corresponds to the logarithm* 0.317879?

Ans. 2.079118.

From the given log. 0.317879

Sub. next less. Table I., 2, 0.301030

0.016849

Next less. Table II., 1039, 0.016616

0.000233

Divide 233 by 417 (tab. diff.), and we obtain 559, nearly, which we annex to 1039, making 1039559, which, multiplied by [illegible] produces 2.079118, the answer.

2. *What number corresponds to the logarithm* 4.031718?

Ans. 10757.673.

As the first figure in the decimal is 0, we had better use Table II. at once.

The number next less in the table is the log. of 1075. As the index is 4, we must have another figure for whole numbers.

Hence, from the given log., 4.031718

Subtract log. of 10750 (Table II.), which is 4.031408

.000310

This difference is less than any other logarithm in Table II. Therefore, we can obtain no other factor, but we can correct 10750.

The next greater logarithm in the table is that of 10760, which is .031812; the tablular diff. is .000404.

Therefore, .000404 : 000310 : : 10 : correction.

Or, 404 : 310 : : 10 : correction = 7.673.

3. *The logarithm of the British gallon is* 2.442909 ; *a cubic inch being the unit. How many cubic inches does it contain?*

Ans. 277.27+

(Art. 151*g*.) Logarithms are exponents; and as powers and roots may be expressed by exponents, therefore, logarithms may be used for finding powers and roots.

To extract the square root of any number, a, we divide its exponent (one understood), by 2, and the result is $a^{\frac{1}{2}}$. To extract the third root, fourth root, &c., we divide the exponent by 3, 4, &c., as the case may be.

The exponent of a number is the logarithm of that number. Hence, to extract roots by logarithms, we have the following rule:

Rule. *Take the logarithm of the number and divide it by 2 for the square root, by 3 for the third root, by 4 for the fourth root, and so on.*

The quotient will be the logarithm of the root sought, and the number corresponding to this logarithm will be the root itself.[*]

(Art. 151*h*.) If $x=a^3$ Then $\log. x=3 \log. a$.

By hypothesis, $x=a.a.a.$

By taking the logarithm of each member, we have

$$\log. x= \log. a + \log. a + \log. a = 3 \log. a. \quad (1)$$

Therefore, in general terms,

$$\text{If} \quad x=a^n \qquad \log. x=n \log. a. (2)$$

Now, let n be a fraction, then $x=a^{\frac{1}{3}}$.

Cube both members; then $x^3=a$. That is, $x.x.x=a$.

By the property of logarithms we have

$$\log. x + \log. x + \log. x = \log. a.$$

$$\text{Or,} \quad 3 \log. x = \log. a.$$

$$\text{Or,} \quad \log. x = \tfrac{1}{3} \log. a. \quad (3)$$

All these equations conform to the rule just given.

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[*] Examples can be taken out of any Arithmetic.

USE AND APPLICATION OF LOGARITHMS.

(Art. 152.) The sciences of trigonometry, mensuration, and astronomy alone, can develop the entire practical utility of logarithms. The science of algebra can only point out their nature, and the first principles on which they are founded. To explain their utility, we must suppose a table of logarithms formed, corresponding to all possible numbers, and by them we may resolve such equations as the following:

1. Given $2^x=10$ to find the value of x.

If the two members of the equation are equal, the logarithms of the two members will be equal, therefore take the logarithm of each member; but as x is a logarithm already, we shall have x log. 2=log. 10.

$$\text{Or } x=\frac{\log.\ 10}{\log.\ 2}=\frac{1}{.30103}=3.3219+$$

2. Given $(729)^{\frac{1}{x}}=3$, to find the value of x.

Raise both members to the x power, and $3^x=729=9^3$,

Or $3^x=3^6$. Hence, $x=6$.

3. Given $a^x+b^y=c$, and $a^x-b^y=d$, to find the values of x and y.

By addition, $2a^x=c+d$. Put $c+d=2m$;

Then $a^x=m$. Take the logarithm of each member, and x log. u=log. m, or $x=\dfrac{\log.\ m}{\log.\ a}$.

By subtracting the second equation from the first and making $c-d=2n$, we shall find $y=\dfrac{\log.\ n}{\log\ b}$.

4. Given $(216)^{\frac{3}{x}}=12$, to find the value of x.

Ans. $x=\dfrac{9\log.\ 6}{\log.\ 12}$

5. Given $\dfrac{ab^x-c}{d}=e$, to find the value of x.

Ans. $x=\dfrac{\log.\ m-\log.\ a}{\log.\ b}$ m being equal to $(de+c)$

6. Given $4^{\frac{x}{3}}=16$, to find the value of x. *Ans.* $x=6$.

7. Given $6^x=\frac{24^6(17)^{\frac{1}{3}}}{71}$ to find the value of x.

Ans. $x=\frac{18 \log.24+\log.17-3 \log.71}{3 \log. 6}$.

8. Required the result of 23.46 multiplied by 7.218, and the product divided by 11.17.

OPERATION.

23.46	. . . .	log. 1.37033
7.218	. . . .	log. 0.85842
	Sum	2.22875
11.17	Subtr.	log. 1.04805
Result, ...	15.16	log. 1.18070.

N. B. The log. of a vulgar fraction is found by subtracting the log. of its denominator from the log. of its numerator.

For instance, $\frac{9}{11}$ is simply 9 divided by 11, and division in logarithms is performed by subtraction.

Thus, from the log. of 9, ... 0.954343
Subtract log. of 11, . . . 1.041393
Log. of $\frac{9}{11}$ therefore is . . . . —1.912850

The decimal part of a logarithm is never minus, but the index is always minus when the number is less than unity. Hence, the logarithm of a very small fraction has a large negative index, and the logarithm of 0 is *minus infinity*. The logarithm of 10 is 1, of 1 is 0, of .1 is —1, of .01 it is —2, of .001 it is —3, and so on. As the decimal number goes down to zero, the index of the logarithm goes *up to minus infinity*.

The previous example was nothing more than obtaining the logarithm of a common fraction, where the numerator consisted of two factors. That example might have been stated thus:

Find the log. of the fraction, $\frac{(23.46)\,(7.218)}{11.17}$

Ans. 1.18070.

CHAPTER V.

COMPOUND INTEREST.

(Art. 153.) Logarithms are of great utility in resolving some questions in relation to compound interest and annuities; but for a full understanding of the subject, the pupil must pass through the following investigation:

Let p represent any principal, and r the interest of a unit of this principal for one year. Then $1+r$ would be the amount of $1, or £1. Put $\mathcal{A}=1+r$.

Now as two dollars will amount to twice as much as one dollar, three dollars to three times as much as one dollar, &c.

Therefore, $1 : \mathcal{A} :: \mathcal{A} : \mathcal{A}^2$=the amount in 2 years,

And $1 : \mathcal{A} :: \mathcal{A}^2 : \mathcal{A}^3$=the amount in 3 years,

&c. &c.

Therefore, $\mathcal{A}^n$ is the amount of one dollar or one unit of the principal in n years, and p times this sum will be the amount for p dollars. Let a represent this amount; then we have this general equation,

$$p\mathcal{A}^n=a.$$

In questions where n, the number of years, is an unknown term, or very large, the aid of logarithms is very essential to a quick and easy solution.

For example, *what time is required for any sum of money to double itself, at three per cent. compound interest?*

Here $a=2p$, and $\mathcal{A}=(1.03)$, and the general equation becomes

$$p(1.03)^n=2p$$

Or $(1.03)^n=2.$ Taking the logarithms

$n \log. (1.03)=\log. 2,$ or $n=\dfrac{\log. 2}{(\log. 1.03)}=\dfrac{.30103}{.012837}=23.45$ years nearly.

2. A bottle of wine that originally cost 20 cents was put away for two hundred years: what would it be worth at the end of that time, allowing 5 per cent. compound interest?

This question makes the general equation stand thus:

(20 cts. being $\frac{1}{5}$ of a dollar) $\frac{1}{5}(1.05)^{200}= a$

Therefore $(1.05)^{200}=5a$

Taking the logarithms . $200 \log. (1.05)=\log. 5+\log. a$

Hence $\log. a=200 \log. (1.05)-\log. 5$. *Ans.* $3458.10

3. A capital of $5000 stands at 4 per cent. compound interest; what will it amount to in 40 years? *Ans.* $24005.10.

4. In what time will $5 amount to $9, at 5 per cent. compound interest? *Ans.* 12.04 years.

5. A capital of $1000 in 6 years, at compound interest, amounted to $1800; what was the rate per cent?

Ans. $\log. (1+r)=\frac{\log. 1.8}{6}$ or $10\frac{3}{10}$ nearly

6. A certain sum of money at compound interest, at 4 per cent. for four years, amounted to 350.95\frac{3}{4}$; what was the sum? *Ans.* $300.

7. How long must $3600 remain, at 5 per cent. compound interest to amount to as much as $5000, at 4 per cent. for 12 years? *Ans.* 16 years, nearly.

ANNUITIES.

(Art. 154.) An annuity is a sum of money payable periodically, for some specified time, or during the life of the receiver. If the payments are not made, the annuity is said to b in *arrear*, and the receiver is entitled to interest on the several payments in arrear.

The *worth* of an annuity in arrear, is the sum of the several payments, together with compound interest on every payment after it became due.

On this definition we proceed to investigate a formula to be applied to calculations respecting annuities.

Let p represent the annual principal or annuity to be paid, and $1+r=\mathcal{A}$, the amount of annuity of principal for one year, at the given rate r.

Let n represent the number of years, and put $\mathcal{A}'$ to represent the entire amount of the annuity in arrear.

It is evident, that on the last payment due, no interest could accrue, and therefore the sum will be p. The preceding payment will have one year's interest; it will therefore be $p\mathcal{A}$; the payment preceding that will have two years' compound interest; and, of course, will be represented by $p\mathcal{A}^2$. (Art. 153.) Hence the whole amount of $\mathcal{A}'$ will be

$$\mathcal{A}'=p+p\mathcal{A}+p\mathcal{A}^2+p\mathcal{A}^3, \text{ \&c., to } p\mathcal{A}^{n-1}.$$

This is a geometrical series, and its sum (Art. 120.) is

$$\mathcal{A}'=\frac{p\mathcal{A}^n-p}{\mathcal{A}-1}=\frac{p[(1+r)^n-1]}{r}$$

This general equation contains four quantities, $\mathcal{A}'$, p, r, and n; any three of them being given in any question, the others can be found, *except* r.

EXAMPLES.

1. An annuity of \$50 has remained unpaid for 6 years, at compound interest on the sums due, at 6 per cent., what sum is now due?

By the general equation,

$$\mathcal{A}'=\frac{50[(1.06)^6-1]}{.06}$$

Taking the log. of both members, we have

$$\log. \mathcal{A}'=\log. 50+\log. [(1.06)^6-1]-\log. .06.$$

The value of $(1.06)^6$, as found by logarithms, is 1.41852, from which subtract 1, as indicated, and take the log. of the decimal number .41852, we then have

$$\log. \mathcal{A}'=1.69897+(-1.62172)-(-2.778151)=2.54218,$$

From which we find, $\mathcal{A}'=\$348.56$ *Ans.*

2. In what time will an annuity of \$20 amount to \$1000, at 4 per cent., compound interest?

The equation applied, we have

$$1000=\frac{20[(1.04)^n-1]}{.04};$$

Dividing by 20, and multiplying by .04, we have

$$2=(1.04)^n-1 \quad \text{or} \quad (1.04)^n=3.$$

$$\textit{Ans.} \quad n=\frac{\log.3}{\log.1.04}=\frac{.477121}{.017033}=28 \text{ years.}$$

3. What will an annuity of $50 amount to, if suffered to remain unpaid for twenty years, at 3½ per cent. compound interest? *Ans.* $1413.98.

4. What is the present value of an annuity or rental of $50 a year, to continue 20 years, discounting at the rate of 3½ per cent., compound interest?

N. B. By question 3d, we find that if the annuity be not paid until the end of 20 years, the amount then due would be $1413.98. If paid now, such a sum must be paid as, put out at compound interest for the given rate and time, will amount to $1413.98.

Now if we had the amount of $1 at compound interest for 20 years, at 3½ per cent., that sum would be to $1 as $1413.98 is to the required *sum*, $710.62.

(Art. 155.) To be more general, let us represent the present worth of an annuity by P. By (Art. 153.) the amount of one dollar for any given rate and time, is A^n; A being $1+r$ and n the number of years. By (Art. 154.) the value of any annuity p remaining unpaid for any given time, n years, at any rate of compound interest r, is $\frac{pA^n-p}{r}$ or A'.

Now by the preceding explanation we may have this proportion:

$$A^n : 1 :: A' : P, \quad \text{or} \quad P=\frac{A'}{A^n} \text{......} (1)$$

Hence, to find the present worth of an annuity, we have this

Rule. *Divide the amount of the annuity supposed unpaid for the given number of years, by the amount of one dollar for the same number of years.*

If in equation (1) we put the value of A, we shall have $PA^n=\frac{pA^n-p}{r}$. Divide both members by A^n, and we have

$$P=\frac{p-\frac{p}{A^n}}{r} \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\; (2)$$

This last equation will apply to the following problems:

5. The annual rent of a freehold estate is p pounds or dollars, to continue forever. What is the present value of the estate, money being worth 5 per cent., compound interest?

Here, as n is infinite, the term, $\frac{p}{A^n}$ becomes 0, and equation (2) becomes $P=\frac{p}{r}=\frac{p}{.05}=20p$; that is, the present value of the estate is worth 20 years' rent.

6. The rent of an estate is \$3000 a year; what sum could purchase such an estate, money being worth 3 per cent., compound interest? *Ans.* \$100000.

7. What is the present value of an annuity of \$350, assigned for 8 years, at 4 per cent.? *Ans.* \$2356.46.

8. A debt due at this time, amounting to \$1200, is to be discharged in seven annual and equal payments; what is the amount of these payments, if interest be computed at 4 per cent.? *Ans.* \$200, nearly.

9. The rent of a farm is \$250 per year, with a perpetual lease. How much ready money will purchase said farm, money being worth 7 per cent. per annum? *Ans.* \$3571$\frac{3}{7}$

10. An annuity of \$50 was suffered to remain unpaid for 20 years, and then amounted to \$1413.98; what was the rate per cent., at compound interest?

N. B. This question is the converse of problem 3, and, of course, the answer must be 3$\frac{1}{2}$ per cent. But the general equation gives us

$$1413.98=\frac{50[(1+r)^{20}-1]}{r},$$

Or $$28.2796=\frac{(1+r)^{20}-1}{r};$$

an equation from which it is practically impossible to obtain r, except by successive approximations.

SECTION VII.

CHAPTER I.

GENERAL THEORY OF EQUATIONS.

(Art. 156.) In (Art. 101.) we have shown that a quadratic equation, or an equation of the second degree, may be conceived to have arisen from the product of two equations of the first degree. Thus, if $x=a$, in one equation, and $x=b$ in another equation, we then have

$$x-a=0,$$

and $$x-b=0;$$

By multiplication, $$x^2-(a+b)x+ab=0.$$

This product presents a quadratic equation, and its two roots are a and b.

If one of the roots be negative, as $x=-a$, and $x=b$, the resulting quadratic is

$$x^2+(a-b)x-ab=0.$$

If both roots be negative, then we shall have

$$x^2+(a+b)x+ab=0.$$

Now let the pupil observe that the exponent of the highest power of the unknown quantity is 2; *and there are two roots. The coëfficient of the first power of the unknown quantity is the algebraic sum of the two roots, with their signs changed; and the absolute term, independent of the unknown quantity, is the product of the roots (the sign conforming to the rules of multiplication).*

When the coefficients and absolute term of a quadratic are not large, and not fractional, we may determine its roots by inspection, by a careful application of these principles

EXAMPLES.

Given $x^2-20x+96=0$, to find x.

The roots must be 12 and 8, for no other numbers will make -20, signs changed, and product 96.

Given $y^2-6y-55=0$ to find y. *Roots* 11 and -5.

Given $x^2-6x-40=0$ to find x. *Roots* 10 and -4.

Given $x^2+6x-91=0$ to find x. *Roots* 7 and -13.

Given $y^2-5y-6=0$ to find y. *Roots* 6 and -1.

Given $y^2+12y-589=0$ to find y.

Here it is not to be supposed that we can decide the values of the roots by inspection; the absolute term is too large; but, nevertheless, the equation has two roots.

Let the roots be represented by P and Q.

From the preceding investigation

$$P+Q=-12 \quad \ldots\ldots\ldots\ldots (1)$$

And
$$PQ=-589 \quad \ldots\ldots\ldots\ldots (2)$$

By squaring eq. (1) $\quad P^2+2PQ+Q^2=\ \ 144$

4 times eq. . . . (2) $\quad 4PQ \quad =-2356$

By subtraction, $\quad P^2-2PQ+Q^2=\ \ 2500$

By evolution, $\quad P-Q=\pm 50$

But $\quad P+Q=-12$

Hence $P=19$ or -31, and $Q=-31$ or $+19$, the true roots of the primitive equation; *and thus we have another method of resolving quadratics.*

(Art. 157.) In the same manner we can show that the product of three simple equations produce a cubic equation, or an equation of the third degree. Conversely, then, an equation of the third degree has three roots.

The three simple equations, $x=a$, $x=b$, $x=c$,[*] may be put in the form of $x-a=0$, $x-b=0$, and $x-c=0$, and the product of these three give

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[*] Of course, x cannot equal different quantities at one and the same time and these equations must not be thus understood.

$$(x-a)(x-b)(x-c)=0\,;$$

and by actual multiplication, we have

$$x^3-(a+b+c)x^2+(ab+ac+bc)x-abc=0.$$

If one of the roots be negative, as $x=-c$, or $x+c=0$, the product or resulting cubic will be

$$x^3-(a+b-c)x^2+(ab-ac-bc)x+abc=0.$$

If two of them be negative, as $x=-b$ and $x=-c$, the resulting cubic will be

$$x^3+(b+c-a)x^2+(bc-ab-ac)x-abc=0.$$

If all the roots be negative, the resulting cubic will be

$$x^3+(a+b+c)x^2+(ab+ac+bc)x+abc=0$$

Every cubic equation may be reduced to this form, and conceived to be formed by such a combination of the unknown term and its roots.

By inspecting the above equations, we may observe

1st. *The first term is the third power of the unknown quantity.*

2d. *The second term is the second power of the unknown quantity, with a coefficient equal to the algebraic sum of the roots, with the contrary sign.*

3d. *The third term is the first power of the unknown quantity, with a coefficient equal to the sum of all the products which can be made, by taking the roots two by two.*

4th. *The fourth term is the continued product of all the roots, with the contrary sign.*

It is easy, then, to form a cubic equation which shall have any three given numbers for its roots.

Assuming x for the unknown quantity, find an equation which shall have 1, 2 and 3 for its roots.

Ans. $x^3-(1+2+3)x^2+(2+3+6)x-6=0\,;$
Or $x^3-6x^2+11x=6.$

Find the equation which shall have 2, 3, and -4 for its roots.

Ans. $x^3-x^2-14x+24=0.$

Find the equation which shall have -3, -4, and $+7$ for its roots. *Ans.* $x^3 \pm 0x^2 - 37x - 84 = 0$.

Or $x^3 - 37x - 84 = 0$.

These four general cases of cubic equations may all be represented by the general form.

Thus: $$x^3 + px^2 + qx + r = 0, \quad \ldots\ldots\ldots\ldots \quad (1)$$

(Art. 158.) When the algebraic sum of three roots is equal to zero, equation (1) takes the form of

$$x^3 + qx + r = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Equation (1) is a regular cubic, and is not susceptible of a direct solution, by Cardan's rule, until it is transformed into another wanting the second term, thus making it take the form of equation (2). To make this transformation, conceive *one* of the roots, or x, in equation (1), represented by $u+v$.

Then
$$\begin{aligned} x^3 &= u^3 + 3u^2v + 3uv^2 + v^3 \\ px^2 &= \quad pu^2 + 2puv + pv^2 \\ qx &= \qquad\qquad qu + qv \\ r &= \qquad\qquad\qquad r \end{aligned}$$

By addition, and uniting the second member according to the powers of u, we shall have

$$u^3 + (3v+p)u^2 + (3v^2 + 2pv + q)u + (v^3 + pv^2 + qv + r) = 0,$$

for the transformed equation. But the object was to make such a transformation that the resulting equation should be deprived of its second power; and to effect this, it is obvious that we must make the coefficient of u^2 equal zero, or $3v+p=0$.

Therefore, $$v = -\tfrac{1}{3}p.$$

Hence, we perceive that if x, in the general equation (1), be put equal to $u - \frac{p}{3}$, there will result an equation in the form of $u^3 + qu + r = 0$, or the form of equation (2).

As $x = u - \frac{p}{3}$, and if a, b, and c represent the roots of equation (1), or the values of x, the roots of (2), or values of u will be

$$a + \tfrac{1}{3}p, \quad b + \tfrac{1}{3}p, \quad \text{and} \quad c + \tfrac{1}{3}p$$

EXAMPLES.

1. Transform the equation $x^3-9x^2+26x-30=0$, into another wanting the second term.

By the preceding investigation, we must assume

$$x=u+3. \quad \text{Here } p=-9\text{; therefore, } -\tfrac{1}{3}p=3.$$

$$\begin{array}{rl}
x^3= & u^3+9u^2+27u+27 \\
-9x^2= & \quad\;\; -9u^2-54u-81 \\
26x= & \qquad\qquad\;\; 26u+78 \\
-30= & \qquad\qquad\qquad\;\; -30 \\
\hline
\text{Sum, } 0= & u^3 \qquad\quad -u-\;6=0,
\end{array}$$

the equation required.

2. Transform the equation $x^3-6x^2+10x-8=0$, into another not containing the square of the unknown quantity.

Put $x=u+2$. *Result*, $u^3-2u-4=0$.

3. Transform $x^3-3x^2+6x-12=0$, into another equation, wanting the second power of the unknown quantity.

Put $x=u+1$. *Result*, $u^3+3u-8=0$.

(Art. 159. We have shown, in the last article, that any regular cubic equation containing all the powers of the unknown quantity can be transformed into another equation deficient of the second power; and hence all cubic equations can be reduced to the form of

$$x^3+3px=2q.$$

We represent the coefficient of x by $3p$, and the absolute term by $2q$, in place of single letters, to avoid fractions, in the course of the following investigation.

Now, if we can find a direct solution to this general equation, it will be a solution of cubic equations generally.

The value of x must be some quantity; and let that quantity, whatever it is, be represented by two parts, $v+y$, or let $x=v+y$. Then the equation becomes

$$(v+y)^3+3p(v+y)=2q.$$

By expanding and reducing, we have

$$v^3+y^3+3(vy+p)(v+y)=2q.$$

Now as we have made an arbitrary division of x into two parts, v and y we can so divide it, that

$$vy+p=0.$$

This hypothesis gives

$$v^3+y^3=2q, \quad \ldots\ldots\ldots\ldots (A)$$

And

$$vy=-p, \quad \ldots\ldots\ldots\ldots (B)$$

Here we have two equations, (A) and (B), containing two unknown quantities, similarly involved, which admit of a solution by quadratics. (Art. 108.) Hence we obtain v and y, and their algebraic sum is x.

From equation (B),

$$y^3=-\frac{p^3}{v^3}.$$

This substituted in equation (A), gives

$$v^3-\frac{p^3}{v^3}=2q;$$

Or, $v^6-2qv^3=p^3$, a quadratic.

Hence $v=\left(q\pm\sqrt{q^2+p^3}\right)^{\frac{1}{3}}$

And $y=\left(q\mp\sqrt{q^2+p^3}\right)^{\frac{1}{3}}$

Or, as $y=\frac{-p}{v}=\frac{-p}{(q\pm\sqrt{q^2+p^3})^{\frac{1}{3}}}$;

Therefore $x=\left(q+\sqrt{q^2+p^3}\right)^{\frac{1}{3}}+\left(q-\sqrt{q^2+p^3}\right)^{\frac{1}{3}} \quad \ldots (C)$

Or, $x=\left(q\pm\sqrt{q^2+p^3}\right)^{\frac{1}{3}}-\frac{p}{\left(q\pm\sqrt{q^2+p^3}\right)^{\frac{1}{3}}} \quad \ldots (D)$

These formulas are familiarly known, among mathematicians, as Cardan's rule.

(Art. 160.) When p is negative, in the general equation, and its cube greater than q^2, the expression $\sqrt{q^2-p^3}$ becomes imaginary; but we must not conclude that the value of x is therefore imaginary; for, admitting the expression $\sqrt{q^2-p^3}$ imaginary, it can be represented by $a\sqrt{-1}$, and the value of x, in equation (C), will be

$$x=(q+a\sqrt{-1})^{\frac{1}{3}}+(q-a\sqrt{-1})^{\frac{1}{3}}$$

Or, $$x=q^{\frac{1}{3}}\left(1+\frac{a}{q}\sqrt{-1}\right)^{\frac{1}{3}}+q^{\frac{1}{3}}\left(1-\frac{a}{q}\sqrt{-1}\right)^{\frac{1}{3}}$$

Or, $$\frac{x}{q^{\frac{1}{3}}}=\left(1+\frac{a}{q}\sqrt{-1}\right)^{\frac{1}{3}}+\left(1-\frac{a}{q}\sqrt{-1}\right)^{\frac{1}{3}}$$

Now by actually expanding the roots of these binomials by the binomial theorem, and adding their results together, the terms containing $\sqrt{-1}$ will destroy each other, and their sum will be a real quantity; and, of course, the value of x will become real. If in any particular case it becomes necessary to make the series converge, change the terms of the binomial, and make $\frac{a}{q}\sqrt{-1}$ stand first, and 1 second.

EXAMPLES.

Given $x^3-6x=5.6$, to find the value of x.

Here, $3p=-6$, and $2q=5.6$, or $p=-2$, and $q=2.8$. Then

$$x=(2.8+\sqrt{7.84-8})^{\frac{1}{3}}+(2.8-\sqrt{7.84-8})^{\frac{1}{3}}, \text{ by equation } (C)$$

Or $$x=(2.8+.4\sqrt{-1})^{\frac{1}{3}}+(2.8-.4\sqrt{-1})^{\frac{1}{3}}$$

Or $$\frac{x}{\sqrt[3]{(2.8)}}=(1+\tfrac{1}{7}\sqrt{-1})^{\frac{1}{3}}+(1-\tfrac{1}{7}\sqrt{-1})^{\frac{1}{3}}$$

Expand the binomials by the binomial theorem, (Art. 135.), and for the sake of brevity, represent $\frac{1}{7}\sqrt{-1}$ by b;

Then $b^2=-\frac{1}{49}$, and $b^4=\frac{1}{49}\times\frac{1}{49}$.

$$(1+\tfrac{1}{7}\sqrt{-1})=1+b. \qquad (1-\tfrac{1}{7}\sqrt{-1})=1-b.$$

$$(1+b)^{\frac{1}{3}}=1+\tfrac{1}{3}b-\frac{2}{3.6}b^2+\frac{2.5}{3.6.9}b^3-\frac{2.5.8}{3.6.9.12}b^4, \text{ \&c.}$$

$$(1-b)^{\frac{1}{3}}=1-\tfrac{1}{3}b-\frac{2}{3.6}b^2-\frac{2.5}{3.6.9}b^3-\frac{2.5.8}{3.6.9.12}b^4, \text{ \&c.}$$

Sum, $$=2-2\left(\frac{2}{3.6}b^2\right)-2\left(\frac{2.5.8}{3.6.9.12}b^4\right)+, \text{ \&c.}$$

$$=2+0.004535-0.000034=2.0045.$$

Therefore,

$$\frac{x}{\sqrt[3]{2.8}}=2.0045 \text{ or } x=(2.0045)(2.8)^{\frac{1}{3}}=2.8256, \text{ nearly}$$

(Art. 161.) Every cubic equation of the form of

$$x^3 \pm px = \pm q$$

has three roots, and their algebraic sum is 0, because the equation is wanting its second term. (Art. 157.)

If the roots be represented by a, b, and c, we shall have $a+b+c=0$, and $abc=\pm q$.

If any two of these roots are equal, as $b=c$, then $a=-2b$ (1), and $ab^2=\pm q$ (2). Putting the value of a taken from equation (1), into equation (2), and we have $-2b^3=\pm q$.

Hence, in case of there being two equal roots, such roots must each equal the cube root of one half the quantity represented by q.

EXAMPLES.

The equation $x^3-48x=128$ has two equal roots; what are he roots?

Here, $-2b^3,=128$, or $b^3=-64$; therefore, $b=-4$.

Two of the roots are each equal to -4, and as the sum of the three roots must be 0, therefore -4, -4, $+8$, must be the three roots.

If the equation $x^3-27x=54$ have two equal roots, what are the roots? *Ans.* -3, -3, and $+6$.

Either of these roots can be taken to verify the equation; and if they do not verify it, the equation has not equal roots.

(Art. 162.) If a cubic equation in the form of

$$x^3+px^2+qx+r=0$$

have two equal roots, each one of the equal roots will be equal to

$$\tfrac{1}{3}(p\pm\sqrt{p^2-3q}).$$

The other root will be twice this quantity subtracted from p, because the sum of the three roots equal p. (Art. 157.)

This expression is obtained from the consideration that the three roots represented by a, b, and c, must form the following equations: (Art. 157.)

$$a+b+c=p \cdots\cdots\cdots (1)$$

$$ab+ac+bc=q \cdots\cdots\cdots (2)$$

$$abc=r \cdots\cdots\cdots (3)$$

On the assumption that two of these roots are equal, that is, $a=b$, equations (1) and (2) become

$$2a+c=p \cdots\cdots\cdots (4)$$

And $$a^2+2ac=q \cdots\cdots\cdots (5)$$

Multiply equation (4) by $2a$, and we have

$$4a^2+2ac=2ap \cdots\cdots\cdots (7)$$

Subtract (5) $$\underline{a^2+2ac=\ \ q} \cdots\cdots\cdots (8)$$

And we have $$3a^2 \qquad =2ap-q.$$

This equation is a quadratic, in relation to the root a, and a solution gives $a=\frac{1}{3}(p\pm\sqrt{p^2-3q})$.

(Art. 163.) A cubic equation in the form of $x^3\pm px=\pm q$ can be resolved as a quadratic, in all cases in which q can be resolved into two factors, m and n, of such a magnitude that $m^2+p=n$.

For the values of p and q, in the general equation, put the assumed values, $mn=q$, and $p=n-m^2$.

Then we have $$x^3+nx-m^2x=mn.$$

Transpose $-m^2x$, and then multiply both members by x, and $x^4+nx^2=m^2x^2+mnx$.

Add $\frac{n^2}{4}$ to both members, and extract square root;

Then $x^2+\frac{n}{2}=mx+\frac{n}{2}$. Drop $\frac{n}{2}$, and divide by x, and $x=m$.

Therefore, if such factors of q can be found, the equation is already resolved, as x will be equal to the factor m.

EXAMPLES.

1. Given $x^3+6x=88$, to find the values of x

Here, $mn=88=4\times22$, $4^2+6=22$. Hence, $x=4$

2. Given $x^3+3x=14$, to find one value of x.

$2\times7=14$, $2^2+3=7$. Hence, $x=2$.

3. Given $x^3+6x=45$, to find one value of x. *Ans.* $x=3$.

4. Given $x^3-13x=-12$, to find x. *Ans.* $x=3$.

5. Given $y^3+48y=104$, to find y. *Ans.* $y=2$.

In the above examples we have given only one answer, or one root; but we have more than once observed, that every equation of the third degree must have three roots. Take, for example, the 4th equation. We have found, as above, one of its roots to be 3. Now we may conceive the equation to have been the product of three factors, one of which was $(x-3)$; therefore the equation must be divisible by $x-3$ without a remainder, (otherwise 3 cannot be a root); and if we divide the equation by $x-3$, the quotient must be the product of the other two factors.

$$\text{Thus,}\quad \begin{array}{rl} x-3) & x^3-13x+12(x^2+3x-4 \\ & x^3-3x^2 \\ \hline & \quad 3x^2-13x \\ & \quad 3x^2-\ \ 9x \\ \hline & \qquad -4x+12 \\ & \qquad -4x+12 \\ \hline \end{array}$$

By putting $x^2+3x-4=0$, and resolving the equation, we find $x=1$, or -4, and the three roots are 1, 3, -4. Their sum is 0, as it should be, as the equation is deficient of its second term.

In the general equation

$$x^3+px^2+qx+r=0.$$

If p and q are each equal to 0, at the same time, the equation becomes $x^3+r=0$, a *binomial equation.*

Every binomial equation has as many roots as there are units in the exponent of the unknown quantity.

Thus $x^3+8=0$, and $x^3-8=0$, or $x^3+1=0$, and $x^3-1=0$, &c., are equations which apparently have but one root, but a full solution will develop three.

Take, for example, $x^3=8$.

By evolution, $x=2$

$$\begin{array}{l}
x-2)x^3-8(x^2+2x+4 \\
\quad\quad\;\; \underline{x^3-2x^2} \\
\quad\quad\quad\quad 2x^2-8 \\
\quad\quad\quad\quad \underline{2x^2-4x} \\
\quad\quad\quad\quad\quad\quad 4x-8 \\
\quad\quad\quad\quad\quad\quad \underline{4x-8}
\end{array}$$

Now by putting $x^2+2x+4=0$, and resolving the equation, we find $x=-1+\sqrt{-3}$, and $x=-1-\sqrt{-3}$; and the three roots of the equation $x^3-8=0$, are 2, $-1+\sqrt{-3}$, and $-1-\sqrt{-3}$, two of them imaginary, but either one, cubed, will give 8.

The three roots of the equation $x^3+1=0$, are

$$-1,\quad \frac{1}{2}+\frac{1}{2}\sqrt{-3},\quad \text{and}\quad \frac{1}{2}-\frac{1}{2}\sqrt{-3}.$$

CHAPTER II.

GENERAL THEORY OF EQUATIONS—CONTINUED.

(Art. 164.) In the last Chapter we confined our investigations to equations of the second and third degrees; and if they are well understood by the pupil, there will be little difficulty, in future, as many of the general properties belong to equations of every degree.

All the higher equations may be conceived to have been formed by the multiplication of the unknown quantity joined to each of the roots of the equation with a contrary sign, as shown in (Art. 157.).

Let a, b, c, d, e, &c., be roots of an equation, and x its unknown quantity, then the equation may be formed by the product of $(x-a)(x-b)(x-c)$, &c., which product we may represent by

$$x^m+Ax^{m-1}+Bx^{m-2}\ldots\ldots Tx+U=0.$$

Now it being admitted that equations can be thus formed by the multiplication of the unknown quantity joined to its roots, *conversely*, when any of its roots can be found, such root, with its contrary sign joined to the unknown term, will form a complete divisor for the equation; and by the division the equation will be reduced one degree, and conversely.

If any quantity, connected to the unknown quantity by the sign plus or minus, divide an equation without a remainder, such a quantity may be regarded as one of the roots of the equation.

The product of all the roots form the absolute term U.

(Art. 165.) *Every equation having unity for the coefficient of the first term, and all the other coefficients, whole numbers, can have only whole numbers for its commensurable* roots.*

This being one of the most important principles in the theory of equations, its enunciation should be most clearly and distinctly understood. Such equations may have *other roots* than whole numbers; but its roots cannot be among the definite and irreducible fractions, such as $\frac{2}{3}$, $\frac{7}{9}$, $\frac{11}{4}$, &c. Its other roots must be among the *incommensurable* quantities, such as $\sqrt{2}$, $(3)^{\frac{1}{3}}$, &c., *i. e.*, surds, indeterminate decimals, or *imaginary* quantities.

To prove the proposition, let us suppose $\frac{a}{b}$ a commensurable but irreducible fraction, to be a root of the equation

$$x^m+Ax^{m-1}+Bx^{m-2}\ldots\ldots Tx+U=0,$$

A, B, &c., being whole numbers.

Substituting this supposed value of x, we have

$$\frac{a^m}{b^m}+A\frac{a^{m-1}}{b^{m-1}}+B\frac{a^{m-2}}{b^{m-2}}\ldots\ldots T\frac{a}{b}+U=0.$$

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* Commensurable numbers are all those that *measure* or can be measured *by* unity; hence, all whole numbers and definite fractions are commensurable.—*Surds*, and imaginary quantities, are incommensurable.

Transpose all the terms but the first, and multiply by b^{m-1}, and we have

$$\frac{a^m}{b}=-\left(Aa^{m-1}+Ba^{m-2}b \ldots Tab^{m-2}+Ub^{m-1}\right)$$

Now, as a and b are prime to each other, b cannot divide a, or any number of times that a may be taken as a factor, for $\frac{a}{b}$ being irreducible, $\frac{a}{b}\times a$ is also irreducible, as the multipliar a will not be measured by the divisor b; therefore $\frac{a^2}{b}$ cannot be expressed in whole numbers. Continuing the same mode of reasoning, $\frac{a^m}{b}$ cannot express whole numbers, but every term in the other member of the equation expresses whole numbers.

Hence, this supposition that the irreducible fraction $\frac{a}{b}$ is a root of the equation, leads to this absurdity, that a series of whole numbers is equal to another quantity *that must contain a fraction.*

Therefore, we conclude that any equation corresponding to these conditions cannot have a definite *commensurable* fraction among its roots.

(Art. 166.) Any equation having fractional coefficients, can be transformed to another in which the coefficients are all whole numbers, and that of the first term unity.

For example, take the equation

$$x^3+\frac{ax^2}{m}+\frac{bx}{n}+\frac{c}{p}=0.$$

Assume $x=\frac{y}{mnp}$, and put this value of x in the equation,

And $$\frac{y^3}{m^3n^3p^3}+\frac{ay^2}{m^3n^2p^2}+\frac{by}{mn^2p}+\frac{c}{p}=0.$$

Multiply every term by $m^3n^3p^3$, and we have

$$y^3+anpy^2+bm^2np^2y+cm^3n^3p^2=0.$$

When m, n, *and* p *have common factors, we may put* x *equal to* y *divided by the least common multiple of these quantities,* as in the following examples:

Transform the equation $x^3+\frac{ax^2}{pm}+\frac{bx}{m}+\frac{c}{p}=0$, into another which shall have no fractional coefficients, and that of the first term unity.

To effect this, it is sufficient to put $x=\frac{y}{pm}$. With this value of x the equation becomes

$$\frac{y^3}{p^3m^3}+\frac{ay^2}{p^3m^3}+\frac{by}{pm^2}+\frac{c}{p}=0.$$

Multiplying every term by p^3m^3, we obtain

$$y^3+ay^2+bp^2my+cp^2m^3=0$$

for the transformed equation required.

Transform the equation $x^4+\frac{5x^3}{6}+\frac{3x^2}{4}+\frac{7x}{24}+\frac{1}{12}=0$, into another, having no fractional coefficients.

Result, $y^4+20y^3+18.24y^2+7(24)^2y+2(24)^3=0.$

(Art. 167.) Now as every commensurable root consists of whole numbers, and as the coefficients are all whole numbers, each term of itself consists of whole numbers, and the commensurable roots are all found among the whole number divisors of the last term; and if these divisors are few and obvious, those answering to the roots of the equation may be found by trial. If the factors are numerous, we must have some systematic method of examining them, such as is pointed out by the following reasoning:

Take the equation $x^4+Ax^3+Bx^2+Cx+D=0.$

Let a represent one of its commensurable roots. Transpose all the terms but the last, and divide every term by a.

$$\frac{D}{a}=-a^3-Aa^2-Ba-C.$$

But, since a is a root of the equation, it divides D without a remainder, the left hand member of this last equation is therefore a whole number, to which transpose C, also a whole number, and represent $\frac{D}{a}+C$ by N.

Then $\quad N=-a^3-Aa^2-Ba.$

Divide each term by a, and transpose B, and we have

$$\frac{N}{a}+B=-a^2-Aa.$$

The right hand member of this equation is an entire quantity, (not fractional), therefore the other member is also an entire quantity; let it be represented by N', and the equation again divided by a.

Then $\quad \frac{N'}{a}=-a-A.$

Transpose $-A$; reasoning the same as before, we can represent the first member by N'', and we then have

$$N''=-a.$$

Divide by a, and $\frac{N''}{a}=-1$. This must be the final result, in case a is a root.

From these operations we draw the following rule for deciding what divisors of the last term are roots of an equation.

Rule. *Divide the last term by the several divisors, (each designated by a,) and add to the quotient the coefficient of the term involving x.*

Divide this sum by the divisors (a), and add to the quotient the coefficient of the term involving x^2.

Divide this sum by the divisors (a), and add to the quotient the coefficient of the term involving x^3.

And thus continue until the first coefficient, A, is transposed, and the sum divided by a; the last quotient will be *minus one*, if a is in fact a root

EXAMPLES.

1. Required the commensurable roots (if any) of the equation

$$x^4+4x^3-x^2-16x-12=0.$$

Divisors	12	6	4	3	2	1	−1	−2	−3	−4	−6	−12
	−12											
Quotients,	−1	−2	−3	−4	−6	−12	12	6	4	3	2	1
Add	−16											
	−17	−18	−19	−20	−22	−28	−4	−10	−12	−13	−14	−15
2d quotients,		−3			−11	−28	4	5	4			
Add		−1										
		−4			−12	−29	3	4	3			
3d quotients,					−6	−29	−3	−2	−1			
Add					+4							
					−2		+1	+2	+3			
4th quotients,					−1		−1	−1	−1			

Now the divisors which correspond to these quotients are 2, −1, −2, −3, and these are, of course, roots of the equation; and, being four of them, and the equation only of the fourth degree, these are all the roots.

2. Required the commensurable roots of the equation $x^3-6x^2+11x-6=0$. *Ans.* 1, 2, 3.

3. Required the commensurable roots of the equation $x^4-6x^2-16x+21=0$. *Ans.* 3 and 1

Here the student might hesitate, as one regular term of the equation is wanting, or rather the coefficient of x^3 is 0. hence, the equation is $x^4\pm 0x^3-6x^2-16x+21=0$.

Go through the form of adding 0.

4. Required the commensurable roots of the equation $x^4-6x^3+5x^2+2x-10=0$. *Ans.* $-1, +5$.

As the commensurable roots are only two, there must be two incommensurable roots; and they can be found by dividing the given equation by $x+1$, and that quotient by $x-5$, and resolving the last quotient as a quadratic.

EQUAL ROOTS.

(Art. 168.) In any equation, as

$$x^5+Ax^4+Bx^3+Cx^2+Dx+E=0,^* \quad \ldots\ldots \quad (1)$$

the roots may be represented by a, b, c, d, e, and either one, put in the place of x, will verify the equation.

Now, let y represent the difference between any two roots, as $a-b$; then $y=a-b$, and by transposition $b+y=a$. But as a will verify the equation, it being a root, its equal, $(b+y)$, substituted for x, will verify it also. That is,

$$(b+y)^5+A(b+y)^4+B(b+y)^3+C(b+y)^2+D(b+y)+E=0.$$

By expanding the powers, and arranging the terms according to the powers of y, we have

$$\left.\begin{array}{l} b^5+5b^4y+10b^3y^2+10b^2y^3+5by^4+y^5 \\ Ab^4+4Ab^3y+\ 6Ab^2y^2+\ 4Aby^3+Ay^4 \\ Bb^3+3Bb^2y+\ 3Bby^2+By^3 \\ Cb^2+2Cby+Cy^2 \\ Db+Dy \\ E \end{array}\right\}=0.$$

---

[*] We might have been more general, and have taken x^m+Ax^{m-1}, &c., for the equation; but, in our opinion, we shall be better comprehended by taking an equation definite in degree: the reasoning is readily understood as general.

Now, as b is a root of the equation, the first column of this transformation is identical with the proposed equation, on substituting the root b for x. Hence, the first column is equal to zero; therefore, let it be suppressed, and the remainder divided by y.

We then have

$$\left.\begin{array}{l} 5b^4 +10b^3y +10b^2y^2+5by^3+y^4 \\ 4Ab^3+6Ab^2y+4Aby^2+Ay^3 \\ 3Bb^2+3Bby+By^2 \\ 2Cb+Cy \\ D \end{array}\right\}=0.$$

On the supposition that the two roots a *and* b *are equal,* y *becomes nothing,* and this last equation becomes

$$5b^4+4Ab^3+3Bb^2+2Cb+D=0.$$

As b is a root of the original equation, x may be written in place of b; then this last equation is

$$5x^4+4Ax^3+3Bx^2+2Cx+D=0 \cdot\cdot\cdot\cdot\cdot\cdot (2)$$

This equation can be *derived* from the primitive equation by the following

RULE. *Multiply each coefficient by the exponent of* x, *and diminish the exponent by unity.*

Equation (2) being derived from equation (1), by the above rule, may be called a derived polynomial.

(Art. 169.) We again remind the reader that b will verify the primitive equation (1), it being a root, and it must also verify equation (2); hence, b at the same time must verify the two equations (1) and (2).

But if b will verify equation (1), that equation is divisible by $(x—b)$, (Art. 164.), and if it will verify equation (2), that equation also, is divisible by $(x—b)$, and $(x—b)$ must be a common measure of the two equations (1) and (2). That is, in case the primitive equation has two roots equal to b.

(Art. 170.) To determine whether any equation contains equal roots, take its derived polynomial by the rule in (Art. 168.), and seek the greatest common divisor (Art. 27.), which designate by

(D),] of the given equation and its first derived polynomial; and if the divisor D is of the first degree, or of the form of $x-h$, then the equation has two roots each equal to h.

If no common measure can be found, the equation contains no equal roots. If D is of the second degree, with reference to x, put $D=0$, and resolve the equation; and if D is found to be in the form of $(x-h)^2$; *then the given equation has three roots equal to* h.

If D be found of the form of $(x-h)(x-h')$, then the given equation has two roots equal to h, and two equal to h'.

Let D be of any degree whatever; put $D=0$, and, if possible, completely resolve the equation; *and every simple root of D is twice a root in the given equation; every double root of D will be three times a root in the given equation*, and so on.

EXAMPLES.

1. Does the equation $x^4-2x^3-7x^2+20x-12=0$ contain any equal roots, and if so, find them?

Its derived polynomial is $4x^3-6x^2-14x+20$.

The common divisor, by (Art. 27.), is found to be $x-2$; therefore, the equation has two roots, equal to 2.

The equation can then be divided *twice* by $x-2$, or *once* by $(x-2)^2$, or by x^2-4x+4. Performing the division, we find the quotient to be x^2+2x-3, and the original equation is now separated into the two factors,

$$(x^2-4x+4)(x^2+2x-3)=0.$$

The equation can now be verified by putting each of these factors equal to zero. From the first we have already $x=2$, and 2, and from the second we may find $x=1$ or -3; hence, the entire solution of the equation gives 1, 2, 2, -3 for the four roots.

2. The equation $x^5+2x^4-11x^3-8x^2+20x+16=0$ has two equal roots; find them. *Ans.* 2 and 2.

3. Does the equation $x^5-2x^4+3x^3-7x^2+8x-3=0$ contain equal roots, and how many?

Ans. It contains *three* equal roots, each equal to 1

4. Find the equal roots, if any, of the equation

$$x^3+x^2-16x+20=0. \qquad \textit{Ans.} \quad 2 \text{ and } 2.$$

5. Find the equal roots of the equation

$$x^4+2x^3-3x^2-4x+4=0.$$

Ans. Two roots equal to 1, and two roots equal -2.

6. Find the equal roots of the equation

$$x^3-5x^2+10x-8=0.$$

Ans. It contains no equal roots.

(Art. 171.) Equations which have no commensurable roots, or those factors of equations which remain after all the commensurable and equal roots are taken away by division, can be resolved only by some method of approximation, if they exceed the third or fourth degree. It is possible to give a direct solution in cases of cubics and in many cases of the fourth degree; but, in practice, approximate methods are less tedious and more convenient.

We may transform any equation into another whose roots shall be greater or less than the roots of the given equation by a given quantity.

Suppose we have the equation

$$x^5+Ax^4+Bx^3+Cx^2+Dx+E=0, \; . \; . \; . \; . \; . \; (1)$$

and require another equation, whose roots shall be less than those of the above by a.

Put $x=a+y$, and, of course, the equation involving y will have roots less than that involving x, by a, because $y=x-a$, or y is less than x, by a.

In place of x, in the above equation, write its equal $(a+y)$ and we have

$$(a+y)^5+A(a+y)^4+B(a+y)^3+C(a+y)^2+D(a+y)+E=0$$

By expanding and arranging the terms according to the powers of y, we shall have, as in (Art. 168.),

$$\left.\begin{array}{r|l|l|l|l}
(a+y)^5=a^5\ +5a^4 & y+10a^3 & y^2+10a^2 & y^3+5a & y^4+y^5 \\
\mathcal{A}(a+y)^4=\mathcal{A}a^4+4\mathcal{A}a^3 & +6\mathcal{A}a^2 & +4\mathcal{A}a & +\mathcal{A} & \ldots \\
B(a+y)^3=Ba^3+3Ba^2 & +3Ba & B & \ldots & \\
C(a+y)^2=Ca^2+2Ca & +C & \ldots & & \\
D(a+y)=Da+D & \ldots\ldots\ldots\ldots & & & \\
E\ldots=E\ldots\ldots\ldots & & & &
\end{array}\right\}\begin{array}{c}=0\\(2)\end{array}$$

After a little observation, these transformations may be made very expeditiously, for the first perpendicular column may be written out by merely changing x to a, in the original equation, and then, *each horizontal column run out by the law of the binomial theorem.*

Thus a^5 becomes $5a^4$, and this, again, $10a^3$, &c.

Now, the first column of the right hand member of this equation consists entirely of known quantities; and the coefficients of the different powers of y are known; hence we have an equation, involving the several powers of y, in form of equation (1),

Or, $y^5+\mathcal{A}'y^4+B'y^3+C'y^2+D'y+E=0$; the equation required; $\mathcal{A}'$, B', &c., representing the known coefficients of the different powers of y.

In commencing this subject, we took an equation definite in degree, for the purpose of giving the pupil more definite ideas; but it is now proper to show the form of transforming an equation of the most general character.

For this purpose, let us take the equation

$$x^m+\mathcal{A}x^{m-1}+Bx^{m-2}\ldots\ldots Gx+H=0.$$

Now let it be required to transform this equation into another whose roots shall be less than the roots of this equation by a.

Put $x=a+y$, as before; then

$$\left.\begin{array}{l|l|l|l}
a^m \quad +ma^{m-1}\cdots\cdot & y+m\frac{(m-1)}{2}a^{m-2} & y^2\cdots\cdot ma & y^{m-1}+y^m \\
Aa^{m-1}+(m-1)Aa^{m-2} & +\frac{(m-1)(m-2)}{2}Aa^{m-3} & \cdots\cdot A & \\
Ba^{m-2}+(m-2)Ba^{m-3} & \cdots\cdots\cdots\cdots & \cdots\cdot & \\
Ca^{m-3}+(m-3)Ca^{m-4} & C & & \\
\cdots\cdots\cdots\cdots & & & \\
Ga \qquad G & & & \\
H & & &
\end{array}\right\}=0$$

(Art. 172.) As the value of a is perfectly arbitrary, we can assume it of such a value as to annihilate either term of the transformed equation, by making the coefficient of the term zero. If it were required to make the y^{m-1} term disappear in the preceding general equation, we must

Put $\quad ma+A=0\cdots\cdots\text{Or}\cdots\cdots a=-\frac{A}{m}$;

That is, in place of putting $x=a+y$, as heretofore, put $\quad x=y-\frac{A}{m}$.

Hence, to transform an equation into another, wanting its second term, put the unknown quantity equal to another unknown quantity, joined to the second coefficient with a contrary sign, divided by the exponent of the degree of the equation.

This principle has already been used in (Art. 100), and in (Art. 158).

If we should desire to make the third term (counting from the highest power of y) of equation (2) to disappear, we must

Put $10a^2+4Aa+B=0$; and this involves the solution of an equation of the second degree, to find the definite value of a. To make the fourth term disappear would require the solution of an equation of the third degree; and so on.

If a is really a root of the primitive equation, then $x=a$, $y=0$, and each perpendicular column of the transformed equation is 0.

If we designate the first perpendicular column of the general transformed equation by X, and the coefficients of the succeeding columns by

$\frac{X'}{1}$, $\frac{X''}{1.2}$, $\frac{X'''}{1.2.3}$, &c., we shall have,

$$X+X'y+\frac{X''}{2}y^2+\frac{X'''}{2.3}y^3 \cdot\cdot\cdot\cdot\cdot\cdot y^m=0.$$

The coefficients of the different powers of y, as X′, X″, X‴, &c. are called *derived polynomials*, because each term of X′ can be derived from the corresponding term of X; and each term of X″ can be derived from the corresponding terms of X′, *by the law of the binomial theorem*, as observed in the first part of this article. But, to recapitulate:

X is derived from the given equation by simply changing x to a.

X′ *is derived from* X *by multiplying each of the terms of* X *by the exponent of* a, *in that term, and diminishing that exponent by unity, and dividing by the exponent of* y *increased by* 1.

X″ *is derived from* X′, *in the same manner as* X′ *is derived from* X; and so on.

X′ is called the first derived polynomial; X″ the second, &c.

To show the utility of this theorem, we propose to transform the following equations:

1. Transform the equation

$$x^4-12x^3+17x^2-9x+7=0,$$

into another, which shall not contain the 3d power of the unknown quantity.

By (Art. 172.), put $x=y+\frac{12}{4}$ $\cdot\cdot\cdot\cdot\cdot$ or $\cdot\cdot\cdot\cdot$ $x=3+y$

Here $a=3$ and $m=4$.

$$X=(3)^4-12(3)^3+17(3)^2-9(3)+7 \cdots \text{or} \cdots X=-110$$

$$X'=4(3)^3-36(3)^2+34(3)-9 \cdots \text{or} \cdots X'=-123$$

$$\frac{X''}{2}=6(3)^2-36(3)+17 \cdots \text{or} \cdots \frac{X''}{2}=-37$$

$$\frac{X'''}{2.3}=4(3)^1-12 \cdots \text{or} \cdots \frac{X'''}{2.3}=\cdot\cdot\ 0$$

Therefore the transformed equation must be

$$y^4-37y^2-123y-110=0$$

2. Transform the equation

$$x^3-6x^2+13x-12=0,$$

into another wanting its second term. Put $x=2+y$.

$$X=(2)^3-6(2)^2+13(2)-12 \cdots \text{or} \cdots X=-2$$

$$X'=3(2)^2-12(2)+13 \cdots \text{or} \cdots X'=+1$$

$$\frac{X''}{2}=3(2)^1-6 \cdots \text{or} \cdots \frac{X''}{2}=0$$

$$\frac{X'''}{2.3}=1 \cdots \text{or} \cdots \frac{X'''}{2.3}=1$$

Therefore the transformed equation must be

$$y^3+y-2=0.$$

3. Transform the equation

$$x^4-4x^3-8x+32=0,$$

into another whose roots shall be less by 2.

Put $x=2+y$. *Result*, $y^4+4y^3-24y=0$.

As this transformed equation has no term independent of y, $y=0$ will verify the equation; and $x=2$ will verify the original equation, and, of course, is a root of that equation.

4. Transform the equation

$$x^4+16x^3+99x^2+228x+144=0,$$

into another whose roots shall be greater by 3.

Put $x=-3+y$. *Result*, $y^4+4y^3+9y^2-42y=0$.

5 Transform the equation

$$x^4-8x^3+x^2+82x-60=0,$$

into another wanting its second term.

Result, $y^4-23y^2+22y+60=0$.

(Art. 173.) *We may transform an equation by division, as well as by substitution, as the following investigation will show.*

Take the equation

$$x^4+Ax^3+Bx^2+Cx+D=0 \quad \ldots\ldots\ldots \quad (1)$$

If we put $x=a+y$, in the above equation, it will be transformed (Art. D.) into

$$y^4+\frac{X'''}{2.3}y^3+\frac{X''}{2}y^2+\frac{X'}{1}y+X=0 \quad \ldots\ldots\ldots \quad (2)$$

As $x=a+y$, therefore $y=x-a$; and put this value of y in equation (2), we have

$$(x-a)^4+\frac{X'''}{2.3}(x-a)^3+\frac{X''}{2}(x-a)^2+X'(x-a)+X=0\ldots(3)$$

Now it is manifest that equation (3) is identical with equation (1), for we formed equation (2) by transforming equation (1), and from (2) to (3) we only reversed the operation.

Now we can divide equation (3), or *in fact equation* (1), by $(x-a)$, and it is obvious that the first remainder will be X.

Divide the quotient, thus obtained, by the same divisor, $(x-a)$, and the second remainder must be X'.

Divide the second quotient by $(x-a)$, and the third remainder must be $\frac{X''}{2}$.

The next remainder must be $\frac{X'''}{2.3}$, &c., &c., according to the degree of the equation

Now if we reserve these remainders, it is manifest that they may form the coëfficients of the required transformed equation; taking the *last* remainder for the *first* coefficient; and so on, in reverse order.

For illustration, let us take the third example of the last article.

$$x=2+y, \quad \text{or} \quad y=x-2.$$

$$\begin{array}{l}
x-2)x^4-4x^3-8x+32(x^3-2x^2-4x-16 \\
\quad x^4-2x^3 \\
\quad \overline{\quad -2x^3-8x} \\
\quad \quad -2x^3+4x^2 \\
\quad \quad \overline{\quad -4x^2-8x} \\
\quad \quad \quad -4x^2+8x \\
\quad \quad \quad \overline{\quad -16x+32} \\
\quad \quad \quad \quad -16x+32 \\
\quad \quad \quad \quad \overline{\quad \quad 0=X}
\end{array}$$

$$\begin{array}{l}
x-2)x^3-2x^2-4x-16(x^2-4 \\
\quad x^3-2x^2 \\
\quad \overline{\quad \quad -4x-16} \\
\quad \quad \quad -4x+\ 8 \\
\quad \quad \quad \overline{\quad -24=X}
\end{array}$$

$$\begin{array}{l}
x-2)x^2-4(x+2 \\
\quad x^2-2x \\
\quad \overline{\quad 2x-4} \\
\quad \quad 2x-4 \\
\quad \quad \overline{\quad 0=\frac{X''}{1.2}}
\end{array}$$

$$\begin{array}{l}
x-2)x+2(1 \\
\quad x-2 \\
\quad \overline{\quad 4=\frac{X'''}{2.3}}
\end{array}$$

Hence the transformed equation is

$$y^4+4y^3\pm0y^2-24y+0=0;$$

or, $y^4+4y^3-24y=0$, as before.

For a further illustration of this method, we will again operate on the first example of the last article.

$$\begin{array}{l}
x-3)x^4-12x^3+17x^2-9x+7(x^3-9x^2-10x-39 \\
\quad \underline{x^4-3x^3} \\
\qquad -9x^3+17x^2 \\
\qquad \underline{-9x^3+27x^2} \\
\qquad\qquad -10x^2-\ 9x \\
\qquad\qquad \underline{-10x^2+30x} \\
\qquad\qquad\qquad -39x+\ 7 \\
\qquad\qquad\qquad \underline{-39x+117} \\
\qquad\qquad\qquad\qquad -110=X. \quad \text{1st Remainder.}
\end{array}$$

$$\begin{array}{l}
x-3)x^3-9x^2-10x-39(x^2-6x-28 \\
\quad \underline{x^3-3x^2} \\
\qquad -6x^2-10x \\
\qquad \underline{-6x^2+18x} \\
\qquad\qquad -28x-39 \\
\qquad\qquad \underline{-28x+84} \\
\qquad\qquad\qquad -123=X'. \quad \text{2d Remainder.}
\end{array}$$

$$\begin{array}{l}
x-3)x^2-6x-28(x-3 \\
\quad \underline{x^2-3x} \\
\qquad -3x-28 \\
\qquad \underline{-3x+\ 9} \\
\qquad\qquad -37=\frac{X''}{2}. \quad \text{3d Remainder.}
\end{array}
\qquad
\begin{array}{l}
x-3)x-3(1 \\
\quad \underline{x-3} \\
\qquad 0=\frac{X'''}{2.3}. \quad \text{4th Rem}
\end{array}$$

Hence $y^4 \pm 0y^3-37y^2-123y-110=0$, must be the transformed equation.

We shall have a 4th remainder, if we operate on an equation of the 4th degree; a 5th remainder with an equation of the 5th degree; and, in general, n number of remainders with an equation of the nth degree.

But to make this method sufficiently practical, the operator must understand

SYNTHETIC DIVISION.

(Art. 174.) Multiplication and division are so intimately blended that they must be explained in connection. For a particular purpose we wish to introduce a particular practical form of performing certain divisions; and to arrive at this end, we commence with multiplication.

Algebraic quantities, containing regular powers, may be multiplied together by using detached coefficients, and annexing the proper literal powers afterwards.

EXAMPLES.

1 Multiply $a^2+2ax+x^2$ by $a+x$.

Take the coefficients. Thus

$$\begin{array}{lrrrr}
 & 1 & +2 & +1 & \\
 & 1 & +1 & & \\ \hline
 & 1 & +2 & +1 & \\
 & & 1 & +2 & +1 \\ \hline
\text{Product, } \ldots\ldots & 1 & +3 & +3 & +1
\end{array}$$

By annexing the powers, we have

$$a^3+3a^2x+3ax^2+x^3$$

2. Multiply x^2+xy+y^2 by x^2-xy+y^2.

As the literal quantities are regular, we may take detached coefficients, thus:

$$\begin{array}{lrrrrr}
 & 1 & +1 & +1 & & \\
 & 1 & -1 & +1 & & \\ \hline
 & 1 & +1 & +1 & & \\
 & & -1 & -1 & -1 & \\
 & & & 1 & +1 & +1 \\ \hline
\text{Product, } \ldots\ldots & 1 & +0 & +1 & +0 & +1
\end{array}$$

Here the second and fourth coefficients are 0; therefore the terms themselves will vanish; and, annexing the powers, we shall have for the full product

$$x^4+x^2y^2+y^4.$$

3. Multiply $3x^2-2x-1$ by $4x+2$.

$$\begin{array}{llll}
3 & -2 & -1 & \\
4 & +2 & & \\
\hline
12 & -8 & -4 & \\
 & 6 & -4 & -2 \\
\hline
12 & -2 & -8 & -2
\end{array}$$

Product, . . . $12x^3-2x^2-8x-2$.

4. Multiply $x^4-ax^3+a^2x^2-a^3x+a^4$ by $x+a$.

$$\begin{array}{llllll}
1 & -1 & +1 & -1 & +1 & \\
1 & +1 & & & & \\
\hline
1 & -1 & +1 & -1 & +1 & \\
 & +1 & -1 & +1 & -1 & +1 \\
\hline
1 & +0 & +0 & +0 & +0 & +1
\end{array}$$

As all the coefficients are zero except the first and last, therefore the product must be

$$x^5+a^5.$$

(Art. 175.) Now if we can multiply by means of *detached coefficients*, in like cases we can divide by means of them.

Take the last example in multiplication, and reverse it, that is, divide x^5+a^5 by $x+a$.

Here we must suppose all the inferior powers of x^5 and a^5 really exist in the dividend, but disappear in consequence of their coefficients being zero; we therefore write *all the coefficients* of the regular powers thus:

$$\begin{array}{llll}
\text{Divisor.} & \text{Dividend.} & & \text{Quotient.} \\
1+1) & 1+0+0+0+0+1 & (& 1-1+1-1+1 \\
 & \underline{1+1} & & \\
 & \quad -1+0 & & \\
 & \quad \underline{-1-1} & & \\
 & \qquad\quad 1+0 & & \\
 & \qquad\quad \underline{1+1} & & \\
 & \qquad\qquad -1+0 & & \\
 & \qquad\qquad \underline{-1-1} & & \\
 & \qquad\qquad\qquad 1+1 & & \\
 & \qquad\qquad\qquad \underline{1+1} & & \\
 & \qquad\qquad\qquad\quad 0 & &
\end{array}$$

Annexing the regular powers to the quotient, we have $x^4-ax^3+a^2x^2-a^3x+a^4$, for the full quotient.

2. Divide $a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5$ by $a^2-2ab+b^2$.

$$\begin{array}{llll}
1-2+1) & 1-5+10-10+5-1 & (& 1-3+3-1 \\
 & \underline{1-2+\ 1} & & \\
 & \quad -3+9-10 & & \\
 & \quad \underline{-3+6-\ 3} & & \\
 & \qquad\quad 3-7+5 & & \\
 & \qquad\quad \underline{3-6+3} & & \\
 & \qquad\qquad -1+2-1 & & \\
 & \qquad\qquad \underline{-1+2-1} & &
\end{array}$$

These coefficients are manifestly the coefficients of a cube; therefore the powers are readily supplied, and are

$$a^3-3a^2b+3ab^2-b^3.$$

N. B. If we change the signs of the coefficients in the divisor, *except* the first, and then *add* the product of those *changed terms*, we shall arrive at the same result.

Perform the last example over again, after changing the signs of the second and third terms of the divisor. Thus,

$$\begin{array}{rrrrrrrl}
1+2-1) & 1 & -5 & +10 & -10 & +5 & -1 & (1-3+3-1 \\
 & 1 & +2 & -1 & & & & \\
\cline{2-4}
\text{Sum}\ \cdot\cdot & * & -3 & +9 & -10 & & & \\
 & & -3 & -6 & +3 & & & \\
\cline{3-5}
\text{Sum}\ \cdot\cdot\cdot\cdot & & * & 3 & -7 & +5 & & \\
 & & & 3 & +6 & -3 & & \\
\cline{4-6}
\text{Sum}\ \cdot\cdot\cdot\cdot\cdot\cdot\cdot & & & * & -1 & +2 & -1 & \\
 & & & & -1 & -2 & +1 & \\
\cline{5-7}
\text{Sum}\ \cdot\cdot\cdot\cdot\cdot\cdot\cdot & & & & * & 0 & +0 &
\end{array}$$

3. Divide $x^3-6x^2+11x-6$ by $x-2$.

Change the sign of the *second* term of the divisor.

$$\begin{array}{rrrrrl}
1+2) & 1 & -6 & +11 & -6 & (1-4+3 \\
 & 1 & +2 & & & \\
\cline{2-4}
 & & -4 & +11 & & \\
 & & -4 & -8 & & \\
\cline{2-5}
 & & & 3 & -6 & \\
 & & & 3 & +6 & \\
\cline{4-5}
 & & & & 0 &
\end{array}$$

Let the reader observe, that when the first figure of the divisor is 1, the first figure of the quotient will be the same as the first figure of the dividend; and the succeeding figures of the quotient are the same as the first figures of the partial dividends.

Now this last operation can be contracted.

Write down the figures of the dividend with their proper signs, and the second figure of the divisor, with its sign changed, on the right. Thus

$$\begin{array}{rrrrl}
1 & -6 & +11 & -6 & (2.=\text{Divisor} \\
 & 2 & -8 & +6 & \\
\hline
(1 & -4 & +3) & 0 &
\end{array}$$

The first figure, 1, *is brought down for the first figure of the quotient.*

The divisor, 2, is put under -6; their sum is -4, which, multiplied by 2, and the product -8 put under the next term, the sum of $+11-8$ is 3, which multiplied by 2, gives 6, and the sum of the last addition is 0, which shows that there is no remainder.

The numbers in the lower line show the quotient, except the last; that shows the remainder, if any.

This last operation is called *synthetic division.*

4. Divide $x^3+2x^2-8x-24$ by $x-3$.

COMMON METHOD.

$$\begin{array}{r}
x-3)x^3+2x^2-8x-24(x^2+5x+7 \\
\underline{x^3-3x^2} \\
5x^2-8x \\
\underline{5x^2-15x} \\
7x-24 \\
\underline{7x-21} \\
-3
\end{array}$$

SYNTHETIC METHOD.

$$\begin{array}{r}
1+2-8-24(3 \\
\underline{3+15+21} \\
(1+5+7)-3
\end{array}$$

Now we are prepared to work the examples in (Art. E.) in a more expeditious manner.

Transform again, the equation $x^4-4x^3-8x+32=0$, to another, whose roots shall be less by 2.

This equation has no term containing x^2, therefore the coefficient of x^2 must be taken $=0$, if we use *Synthetic Division.*

FIRST OPERATION.

$$\begin{array}{r}
1-4\pm0-8+32\ (2 \\
\underline{2-4-8-32} \\
(1-2-4-16),\ 0=X.
\end{array}$$

SECOND OPERATION.

$$\begin{array}{r}
1-2-4\ -16\ (2 \\
\underline{2\pm0\ -\ 8} \\
(1+0-4),-24=X'.
\end{array}$$

THIRD OPERATION.

$$\begin{array}{rrl}
1\pm0 & -4 & (2 \\
2 & +4 & \\
\hline
(1+2) & 0=\frac{X''}{2}. &
\end{array}$$

FOURTH OPERATION.

$$\begin{array}{rl}
1+2 & (2 \\
2 & \\
\hline
(1) & 4=\frac{X'''}{2.3}.
\end{array}$$

Hence our transformed equation is $y^4+4y^3-24y=0$, as before.

To transform an equation of the fourth degree, we must have four operations in division; an equation of the nth degree n operations, as before observed.

But these operations may be all blended in one. Thus

$$\begin{array}{rrrrrl}
1 & -4 & \pm0 & -8 & 32 & (2 \\
 & 2 & -4 & -8 & -32 & \\
\cline{2-5}
 & -2 & -4 & -16 & 0 & =X \\
 & 2 & 0 & -8 & & \\
\cline{2-4}
 & 0 & -4 & -24 & =X & \\
 & 2 & +4 & & & \\
\cline{2-3}
 & 2 & 0=\frac{X''}{2} & & & \\
 & 2 & & & & \\
\cline{2-2}
 & 4=\frac{X'''}{2.3}. & & & &
\end{array}$$

We omit the first column, except in the first line, as there are no operations with it.

The pupil should observe the structure of this operation. It is an equation of the 4th degree, and there are four sums in addition, in the 2d column; three in the next; two in the next, &c., giving the whole a diagonal shape.

Transform the equation $x^4-12x^3+17x^2-9x+7=0$, into another whose root shall be 3 less.

OPERATION.

$$\begin{array}{rrrrrl}
1 & -12 & +17 & -9 & +7 & (3 \\
 & +3 & -27 & -30 & -117 & \\
\cline{2-5}
 & -9 & -10 & -39 & -110=X & \\
 & +3 & -18 & -84 & & \\
\cline{2-4}
 & -6 & -28 & -123=X' & & \\
 & +3 & -9 & & & \\
\cline{2-3}
 & -3 & -37=\frac{X''}{2} & & & \\
 & 3 & & & & \\
\cline{2-2}
 & 0=\frac{X'''}{2.3}. & & & &
\end{array}$$

Hence the transformed equation is

$$y^4+0y^3-37y^2-123y-110=0.$$

Transform the equation $x^3-12x-28=0$, into another whose roots shall be 4 less.

$$\begin{array}{rrrrl}
1 & 0 & -12 & -28 & (4 \\
 & 4 & +16 & +16 & \\
\cline{2-4}
 & 4 & 4 & -12=X & \\
 & 4 & 32 & & \\
\cline{2-3}
 & 8 & 36=X' & & \\
 & 4 & & & \\
\cline{2-2}
 & 12=\frac{X''}{2}. & & &
\end{array}$$

Hence the transformed equation must be $y^3+12y^2+36y-12=0$, on the supposition that we put $y=x-4$.

Transform the equation $x^3-10x^2+3x-6946=0$, into another whose roots shall be less by 20.

Put . . . . $x=20+y$.

$$\begin{array}{rrrrl}
1 & -10 & 3 & -6946 & (20 \\
 & \underline{20} & \underline{200} & \underline{4060} & \\
 & 10 & 203 & \underline{\underline{-2886}} & \\
 & \underline{20} & \underline{600} & & \\
 & 30 & \underline{\underline{803}} & & \\
 & \underline{20} & & & \\
 & \underline{\underline{50}} & & &
\end{array}$$

The three remainders are the numbers just above the *double lines*, which give the following transformed equation:

$$y^3+50y^2+803y-2886=0.$$

Transform this equation into another, whose roots shall be less by 3. Put $y=3+z$.

$$\begin{array}{rrrrl}
1 & 50 & 803 & -2886 & (3 \\
 & \underline{3} & \underline{159} & \underline{+2886} & \\
 & 53 & 962 & \underline{\underline{0}} & \\
 & \underline{3} & \underline{168} & & \\
 & 56 & \underline{\underline{1130}} & & \\
 & \underline{3} & & & \\
 & \underline{\underline{59}} & & &
\end{array}$$

Hence the second transformed equation is

$$z^3+59z^2+1130z=0.$$

This equation may be verified by making $z=0$; which gives

$$y=3 \quad \text{and} \quad x=20+3=23.$$

Thus we have found the exact root of the original equation by successive transformations; and on *this principle* we shall hereafter give a general rule to approximate to incommensurable roots of equations of any degree; but before the pupil can be prepared to comprehend and surmount every difficulty, he must pay more attention to general theory, as developed in the following Chapter

CHAPTER III.

GENERAL PROPERTIES OF EQUATIONS.

(Art. 176.) *Any equation, having only negative roots, will have all its signs positive.*

If we take $-a$, $-b$, $-c$, &c., to represent the roots of an equation, the equation itself will be the product of the factors;

$$(x+a),\ (x+b),\ (x+c),\ \&c.,\ =0:$$

and it is obvious that all its signs must be positive.

From this, we decide at once, that the equation $x^4+3x^3+6x+6=0$; or any other numeral equation, having *all* its signs plus, can have no rational positive roots.

(Art. 177.) *Surds, and imaginary roots, enter equations by pairs.*

Take any equation, as

$$x^4+Ax^3+Bx^2+Cx+D=0,$$

and suppose $(a+\sqrt{b})$ to be one of its roots, then $(a-\sqrt{b})$ must be another.

In place of x, in the equation, write its equal, and we have

$$(a+\sqrt{b})^4+A(a+\sqrt{b})^3+B(a+\sqrt{b})^2+C(a+\sqrt{b})+D=0$$

By expanding the powers of the binomial, we shall find some terms rational and some surd. The terms in which the odd powers of $\sqrt{b}$ are contained will be surd; the other terms rational; and if we put R to represent the rational part of this equation, and $S\sqrt{b}$ to represent the surd part, then we must have

$$R+S\sqrt{b}=0.$$

But these terms not having a common factor throughout, cannot equal 0, unless we have separately $R=0$, and $S=0$; and if this be the case we may have

$$R-S\sqrt{b}=0.$$

This last equation, then, is one of the results of $(a+\sqrt{b})$, being a root of the equation

Now if we write $(a-\sqrt{b})$ in place of x, in the original equation, and expand the binomials, using the same notation as before, we shall find

$$R-S\sqrt{b}=0.$$

But we have previously shown that this equation must be true; and any quantity, which, substituted for x, reduces an equation to zero, is said to be a root of the equation; therefore $(a-\sqrt{b})$ is a root.

The same demonstration will apply to $(+\sqrt{a})$, $(-\sqrt{a})$, to $+\sqrt{-a}$, $-\sqrt{-a}$, and to imaginary roots in the form of

$$(a+b\sqrt{-1}),\quad (a-b\sqrt{-1}).$$

(Art. 178.) *If we change the signs of the alternate terms of an equation, it will change the signs of all its roots.*

At first, we will take an equation of an even degree.

If a is a root to the equation

$$x^4+Ax^3+Bx^2+Cx+D=0 \quad \ldots\ldots\ldots \quad (1)$$

then will $-a$ be a root to the equation

$$x^4-Ax^3+Bx^2-Cx+D=0 \quad \ldots\ldots\ldots \quad (2)$$

Write a for x, in equation (1), and we have

$$a^4+Aa^3+Ba^2+Ca+D=0 \quad \ldots\ldots\ldots \quad (3)$$

Now write $-a$ for x, in equation (2), and we have

$$a^4+Aa^3+Ba^2+Ca+D=0 \quad \ldots\ldots\ldots \quad (4)$$

Equations (3) and (4) are identical; therefore if a, put for x in equation (1), gives a true result, $-a$ put for x in equation (2), gives a result equally true.

We will now take an equation of an odd degree.

If the equation $\quad x^3+Ax^2+Bx+C=0,$

have a for a root, then will the equation

$$x^3-Ax^2+Bx-C=0,$$

have $-a$ for a root.

From the first $\quad a^3+Aa^2+Ba+C=0.$

From the second $\quad -a^3-Aa^2-Ba-C=0.$

This second equation is identical with the first, if we change *all* its signs, which does not essentially change an equation.

The equation $x^4+x^3-19x^2+11x+30=0$, has -1, 2, 3, and -5, for its roots; then from the preceding investigation we learn that the equation

$$x^4-x^3-19x^2-11x+30=0,$$

must have 1 -2, -3, and $+5$ for its roots.

(Art. 179.) *If we introduce one positive root into an equation, it will produce at least one variation in the signs of its term; if two positive roots, at least two variations.*

The equation $x^2+x+1=0$, having no variation of signs, can have no positive roots. (Art. 176.) Now if we introduce the root $+2$, or which is the same thing, multiply by the factor $x-2$,

$$\begin{array}{llll}
x^2+ & x+1 & & \\
x-2 & & & \\
\hline
x^3+ & x^2+ & x & \\
 & -2x^2 & -2x & -2 \\
\hline
\end{array}$$

$$\text{Then}\ldots\ldots x^3-x^2-x-2=0;$$

and here we find one *variation* of signs from $+x^3$ to $-x^2$, and one *permanence* of signs through the rest of the equation.

If we take this last equation and introduce another positive root, say $+5$, or multiply it by $x-5$, we shall then have

$$\begin{array}{rrrrr}
1-1 & -1 & -2 & & \\
1-5 & & & & \\
\hline
1-1 & -1 & -2 & & \\
-5 & +5 & +5 & +10 & \\
\hline
\end{array}$$

$$x^4-6x^3+4x^2+3x+10=0.$$

Here are *two* variations of signs, *one* from $+x^4$ to $-6x^3$, and another from $-6x^3$ to $+4x^2$.

And thus we might continue to show that every positive root, introduced into an equation, will produce at least one variation of signs. But we must not conclude that the converse of this proposition is true.

Every positive root will give one variation of signs; but every variation of signs does not necessarily show the existence of a positive root.

For an equation may have

$$(a+b\sqrt{-1}),\quad (a-b\sqrt{-1}),\quad -c,\quad -d,$$

for roots; then the equation will be expressed by the product of the factors

$$(x^2-2ax+a^2+b^2)\ (x+c)\ (x+d)=0.$$

As one of these terms, $(-2ax)$, has the minus sign, it will produce some *minus* terms in the product; and there must necessarily be *variations* of signs; yet there is no positive root. At the same time, the whole factor in which the *minus* term is found, must be *plus*, whatever value be given to x, as it is evidently equal to $(x-a)^2+b^2$, the sum of two squares.

The equation

$$x^4-2x^3-x^2+2x+10=0,$$

has two *variations* of signs, and two *permanences*, but the roots are all *imaginary*, viz.,

$$(2+\sqrt{-1}),\ (2-\sqrt{-1}),\ (-1+\sqrt{-1}), \text{ and } (-1-\sqrt{-1}).$$

If it were not for imaginary roots, the number of variations among the signs of an equation would indicate the number of plus roots: and this number, taken from the degree of the equation, would leave the number of negative roots; or the number of permanences of signs would at once show the number of negative roots.

To determine *a priori* the number of real roots contained in any equation, has long baffled the investigations of mathematicians; and the difficulty was not entirely overcome until 1829, when M. Sturm sent a complete solution to the French Academy. The investigation is known as Sturm's Theorem, and will be presented in the following Chapter.

LIMITS TO ROOTS.

(Art. 180.) All positive roots of an equation are comprised between zero and infinity; and all negative roots between zero and *minus* infinity; but it is important to be able *at once* to assign much narrower limits.

We have seen, (Art. 179.), that every equation, having a positive root, must have at least one variation among its signs, and at least one minus sign.

If the highest power is minus, change all the signs in the equation.

Now we propose to show that the greatest positive root must be less than the greatest negative coefficient plus one.

Take the equation

$$x^4+Ax^3+Bx^2+Cx+D=0.$$

It is evident, that as the first term must be positive for all degrees, x must be greater and greater, as more of the other terms are minus: then x must be greatest of all when all the other terms are minus, and each equal to the greatest coefficient, (D being considered the coefficient of x^0).

Now as A, B, &c., are supposed equal, and all minus, we shall have

$$x^4-A(x^3+x^2+x+1)=0.$$

For the first trial take $x=A$, and transpose the minus quantity, and we have

$$A^4=A(A^3+A^2+A+1);$$

Divide by A^4, and we have

$$1=1+\frac{1}{A}+\frac{1}{A^2}+\frac{1}{A^3}.$$

Now we perceive that the second member of the equation is greater than the first, and the expression is not, in fact, an equation. $x=A$ proves x not to be large enough.

For a second trial put $x=A+1$.

Then $(A+1)^4=A[(A+1)^3+(A+1)^2+(A+1)+1]$

Dividing by $(A+1)^4$, we have

$$1=\frac{A}{A+1}+\frac{A}{(A+1)^2}+\frac{A}{(A+1)^3}+\frac{A}{(A+1)^4}.$$

We retain the sign of equality for convenience, though the members are not equal. The second member consists of terms in geometrical progression, and their sum, (Art. 120), is $1-\frac{1}{(A+1)^4}$. Hence the first member is greater than the second, which shows that $(A+1)$ substituted for x, is too great. But A was too small, therefore the real value of x, in the case under consideration, must be more than A and less than $(A+1)$.

That is, the greatest positive root of an equation, in the most extreme case, must be less than the greatest negative coefficient plus one.

In common cases the limit is much less.

From this, we at once decide that the greatest positive root of the equation $x^5-3x^4+7x^3-8x^2-9x-12=0$, is less than 13.

Now change the second, and every alternate sign, and we have the equation

$$x^5+3x^4+7x^3+8x^2-9x+12=0.$$

The greatest positive root, in this equation, is less than 10; but, by (Art. 178.), the greatest positive root of this equation is the greatest negative root of the preceding equation; therefore 10 is the greatest limit of the negative roots of the first equation; and all its roots must be comprised between +13 and —10; but as this equation does not present an extreme case, the coefficients after the first are not *all* minus, nor equal to each other; therefore the *real* limits of its roots must be much within +13 and —10. In fact, the greatest positive root is between 3 and 4, and the greatest negative root less than 1.

If it were desirable to find the limits of the least root, put $x=\frac{1}{y}$, and transform the equation accordingly. Then find, as just directed, the *greatest limit* of y, in its equation; which will, of course, correspond to the least value of x in its equation.

(Art. 181.) *If we substitute any number less than the least root, for the unknown quantity, in any equation of an even degree,* THE RESULT WILL BE POSITIVE. *And if the degree of the equation be odd,* THE RESULT WILL BE NEGATIVE.

Let a, b, c, &c., be roots of an equation, and x the unknown quantity. Also, conceive a to be the least root, b the next greater, and so on. Then the equation will be represented by

$$(x-a)(x-b)(x-c)(x-d), \text{ \&c., } =0.$$

Now in the place of x substitute any number h less than a, and the above factors will become

$$(h-a)(h-b)(h-c)(h-d), \text{ \&c.}$$

Each factor essentially negative, and the product of an *even* number of negative factors, is positive; and the product of an odd number is negative; therefore our proposition is proved.

Scholium.—If we conceive h to increase continuously, until it becomes equal to a, the first factor will be zero; and the product of them all, whether odd or even, will be zero, and the equation will be zero, as it should be when h becomes a root.

If h *increases and becomes greater than* a, *without being equal to b, the result of substituting it for* x *will be* NEGATIVE, *in an equation of an even degree, and positive in an equation of an odd degree.*

For in that case the first factor will be *positive*, and all the other factors negative; and, of course, the signs of their product will be alternately *minus* and *plus*, according as an even or odd number of them are taken.

If h is conceived to increase until it is equal to b, then the second factor is zero, and its substitution for x will verify the equation. If h becomes greater than b, and not equal to c, then the first two factors will be positive; the rest negative; and the result of substituting h for x will give a positive or negative result, according as the degree of the equation is even or odd.

If we conceive h to become greater than the greatest root, then all the factors will be positive, and, of course, their product positive.

For example, let us form an equation with the four roots -5, 2, 6, 8, and then the equation will be

$$(x+5)(x-2)(x-6)(x-8)=0,$$

$$\text{Or} \dots \; x^4-11x^3-4x^2+284x-480=0.$$

(The greater a *negative* number is, the less it is considered.)

Now if we substitute -6 for x, in the equation, the result must be positive. Let -6 increase to -5, and the result will be 0. Let it still increase, and the result will be negative, until it has increased to $+2$, at which point the result will again be 0.

If we substitute a number greater than 2, and less than 6, for x, in the equation, the result will again be *positive*. A number between 6 and 8, put for x, will render the equation *negative*; and a number more than 8 will render the equation positive; and if the number is still conceived to increase, there will be no more change of signs, *because we have passed all the roots.*

If in any equation we substitute numbers for the unknown quantity, which differ from each other by a less number than the difference between any two roots, and commence with a number less than the least root, and continue to a number greater than the greatest root, we shall have as many changes of signs in the results of the substitution as the equation has real roots.

If *one* real root lies between two numbers substituted for the unknown quantity, in any equation, the results will necessarily show a *change of signs.*

If *one*, or *three*, or any *odd* number of roots, lie between the two numbers substituted, the results will show a *change of signs*

If an *even number of roots* lie between the two numbers substituted, the results will show *no* change of signs.

In the last equation, if we substitute -6 for x, the result will be *plus.*

If we substitute $+3$, the result will also be *plus*, and give no indication of the two roots -5 and $+2$, which lie between.

(Art. 182.) If an equation contains *imaginary* roots, the factors pertaining to such roots will be either in the form of (x^2+a), or in the form of $[(x-a)^2+b^2]$, both positive, whatever numbers may be substituted for x, either positive or negative; hence, if no other than imaginary roots enter the equation, all substitutions for x will give positive results, and of course, no changes of sign. *It is only when the substitutions for* x *pass real roots that we shall find a change of signs.*

CHAPTER IV.

GENERAL PROPERTIES OF EQUATIONS—CONTINUED.

(Art. 183.) If we take any equation which has all its roots real and unequal, and make an equation of its first derived polynomial, the least root of this derived equation will be greater than the least root of the primitive equation, and less than the next greater.

If the primitive equation have equal roots, the same root will verify the derived equation.[*]

We shall form our equations from known *positive roots.*

Let a, b, c, d, &c., represent roots; and suppose a less than b, b less than c, &c., and x the unknown quantity. An equation of the second degree is

$$x^2-(a+b)x+ab=0;$$

Its first derived polynomial is

$$2x-(a+b).$$

If we make an equation of this, that is, put it equal to 0, we shall have

$$x=\frac{a+b}{2}.$$

Now if b is greater than a, the value of x is more than a, and less than b, and proves our proposition for all equations of the second degree. If we suppose $a=b$, then $x=a$, in both equations.

An equation of the third degree is

$$x^3-(a+b+c)x^2+(ab+ac+bc)x-abc=0 \; . \; . \; . \; . \; . \; (1)$$

Its first derived polynomial is

$$3x^2-2(a+b+c)x+ab+ac+bc=0 \; . \; . \; . \; . \; . \; . \; (2)$$

This equation, being of the second degree, has two roots, and *only* two.

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[*] To ensure perspicuity and avoid too abstruse generality, we operate on equations definite in degree; the result will be equally satisfactory to the learner, and occupy, comparatively, but little space.

Now if we can find a quantity which, put for x, will verify equation (2), that quantity must be one of its roots. If we try two quantities, and find a change of signs in the results, we are sure a root lies between such quantities. (Art. 181.) Therefore we will try a, or write a in place of x. As b and c are each greater than a, we will suppose that

$$a=a$$
$$b=a+h$$
$$c=a+h'.$$

With these substitutions, equation (2) becomes

$$3a^2-2(3a+h+h')a+3a^2+2ah+2ah'+hh'=0;$$

Reduced, gives . . . . . . $+hh'=0$.

Therefore a cannot be a root; if it were we should have $0=0$.

If we now make a substitution of b for x, or rather $(a+h)$ for x, and reduce the equation, we shall find

$$(h-h')h=0.$$

It is apparent that this quantity is essentially *minus*, as h' is greater than h. Hence, as substituting a for x, in the equation, gives a small *plus* quantity, and b for x gives a small *minus* quantity, therefore *one* value of x, to verify equation (2), must lie between a and b.

This proves the proposition for equations of the third degree; and in this manner we may prove it for any degree; but the labor of substituting for a high equation would be very tedious.

If we suppose $a=b$, and put $c=a+h'$, and then substitute a in place of x, we shall find equations (1) and (2) will be verified.

Therefore in the case of equal roots, the equation and its first derived polynomial will have a common measure, as before shown in (Art. 168).

If all the roots of an equation are equal, the equation itself may be expressed in the form of

$$(x-a)^m=0.$$

Its first derived polynomial, put into an equation, will be

$$m(x-a)^{m-1}=0.$$

It is apparent that the primitive equation has m roots equal to a; and the derived equation, $(m-1)$, roots also equal to a.

Lastly, take a general equation, as

$$x^m+Ax^{m-1}+Bx^{m-2}\ldots\ldots Rx+S=0.$$

Its first derived polynomial, taken for an equation, will be

$$mx^{m-1}+(m-1)Ax^{m-2}\ldots\ldots R=0.$$

We may suppose this general equation composed of the factors

$$(x-a)(x-b)(x-c),\ \&c.,\ =0,$$

and also suppose b greater, ***but insensibly greater***, than a; c insensibly greater than b, &c. Then the equation will be nearly

$$(x-a)^m=0\,;$$

and its derived polynomial,

$$m(x-a)^{m-1}=0,$$

cannot have a ***root less than*** (a), the least root of the primitive equation; but its root cannot equal a, unless the primitive equation have equal roots; therefore it must be greater.

By the same mode of reasoning we can show that the greatest root of an equation is greater than the greatest root of its derived equation; hence the roots of the derived equation are intermediate, in value, to the roots of the primitive equation, or contained within narrower limits.

(Art. 184.) *If we take any equation, not having equal roots, and consider its first derived polynomial also an equation, and then substitute any quantity less than the least root of either equation, for the unknown quantity, the result of such substitution will necessarily give opposite signs.*

Let a, b, c, &c., represent the roots of a primitive equation, and a', b', &c., roots of its first derived polynomial; x the unknown quantity. Then the equation will be

$$(x-a)(x-b)(x-c),\ \&c.,\ \text{to } m \text{ factors } =0\,;$$

the derived equation will be

$$(x-a')(x-b')(x-c'),\ \&c.,\ \text{to } (m-1) \text{ factors } =0.$$

Now if we substitute h for x, and suppose h less than either root, then every factor, in both equations, will be *negative.*

The product of an *even* number of negative factors is *positive* and the product of an *odd* number is *negative;* and if the factors, in the primitive equation are even, those in the derived equation must be odd.

Hence any quantity less than any root of either equation, will necessarily give to these functions opposite signs.

(Art. 185.) Now if we conceive h to increase until it becomes equal to a, the least root, the factor $(x—a)$ will be 0, and reduce the whole equation to 0. Let h still increase and become greater than a, and not equal to a', (which is necessarily greater than a, (Art. 184.), and the factor $(x—a)$ will become *plus*, while all the other factors, in both equations, will be *minus*, and, of course, leave the same number of *minus* factors in both functions, which must give them the same sign. Consequently, in passing the least root of the primitive equation a *variation* is changed into a permanence.

Sturm's Theorem.

(Art. 186.) If we take any equation not having equal roots, and its first derived polynomial, and operate with these functions as though their common measure was desired, reserving the several remainders with their signs changed, and make equations of these functions, namely, the primitive equation, its first derived polynomial and the several remainders with their signs changed, and then substitute any assumed quantity, h, for x, in the several functions, noting the variation of signs in the result; afterwards substitute another quantity, h', for x, and again note the variation of signs; *the difference in the number of variation of signs, resulting from the two substitutions, will give the number of real roots between the limits* h *and* h'.

If *$-\infty$ and $+\infty$ are taken for h and h', we shall have the whole number of real roots; which number, subtracted from the degree of the equation, will give the number of imaginary roots.

DEMONSTRATION.

Let X represent an equation, and X′ its first derived polynomial.

In operating as for common measure, denote the several quotients by Q, Q' Q'', &c., and the several remainders, with their *signs changed*, by R, R', R'', &c.

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* Symbols of infinity.

In these operations, be careful not to strike out or introduce *minus* factors, as they change the signs of the terms; then a *re-change* of signs in the remainder would be erroneous.

From the manner of deriving these functions, we must have the following equations.

$$\frac{X}{X'}=Q-\frac{R}{X'}$$

$$\frac{X'}{R}=Q'-\frac{R'}{R}$$

$$\frac{R}{R'}=Q''-\frac{R''}{R'}$$

$$\frac{R'}{R''}=Q'''-\frac{R'''}{R''}$$

&c. &c.

Clearing these equations of fractions, we have

$$\left.\begin{aligned}X&=X'Q-R\\X'&=RQ'-R'\\R&=R'Q''-R''\\R'&=R''Q'''-R'''\\&\\R_{m-2}&=R_{m-1}Q_m-R_m\end{aligned}\right\}\ldots(A)$$

As the equation $X=0$ must have no equal roots, the functions X and X' can have no common measure (Art. 168.), and we shall arrive at a final remainder, independent of the unknown quantity, and not zero.

Proposition 1. *No two consecutive functions, in equations* (A), *can become zero at the same time.*

For, if possible, let such a value of h be substituted for x, as to render both X' and R zero at the same time; then the second equation of (A) will give $R'=0$. Tracing the equations, we must finally have the last remainder $R_m=0$; but this is inadmissible; therefore the proposition is proved.

Prop. 2. *When one of the functions becomes zero, by giving* *a particular value to* x, *the adjacent functions between which it is placed must have opposite signs.*

Suppose R′ in the third equation, (A), to become 0, then the equation still existing, we must have $R=-R''$.

The truth of Sturm's Theorem rests on the facts demonstrated in Arts. 184, 185 and in the two foregoing propositions.

If we put the functions X, X′, R, R′, &c., each equal to 0; that is, make equations of them, and afterwards substitute any quantity for x, in these functions, less than any root, the first and second functions, X and X′, will have opposite signs, (Art. 184.); and the last function will have a sign independent of x, and, of course, invariable for all changes in that quantity. The other functions may have either *plus* or *minus*, and the signs have a certain number of variations.

Now all changes in the number of these variations must come through the variations of the signs in the primitive function X. *A change of sign in any other function will produce no change in the number of variations in the series.*

For, conceive the following equations to exist:

$$\left.\begin{array}{l} X=0 \\ X'=0 \\ R=0 \\ \&c.\&c. \end{array}\right\}\cdots(B)$$

Now take $x=h$, yet h really less than any root of the equations, (B), and we may have the following series of signs:

$$\left.\begin{array}{l} X=- \\ X'=+ \\ R=- \\ R'=- \\ R''=- \\ R'''=+ \end{array}\right\}\cdots(C)$$

Or we may have any other order of signs, *restricted* only to the fact that the signs of the two first functions must be opposite, and the last invariable, or unaffected by all future substitutions.

Here are three variations of signs.

Now conceive h to increase. No change of signs can take place in any of the equations, *unless* h becomes equal and passes a root of that equation; and as there are no equal roots, no two of these functions can become 0 at the same time (Prop. 1.); hence a change of sign of one function does not permit a change in another; therefore by the increase of h, one of the functions, (C), will become 0, and a further increase of h will change its sign.

In the series of signs as here represented, X′ cannot be the first to change signs, for that would leave the adjacent functions, X and R, of the same sign, contrary to Prop. 2; nor can the function R′ be the first to change sign, for the same reason.

Hence X or R or R″ must be first to change sign.

If we suppose X to change sign, the other signs remaining the same, the number of variations of signs is reduced by unity.

If R or R″ change sign, the number of variations cannot be changed; a permanence may be made or reduced, and all cases that can happen with three consecutive functions may be expressed by the following combinations of signs;

$$\begin{array}{lccc} & + & \pm & - \\ \text{Or} & - & \pm & + \end{array}\Big\}$$

either of which gives one variation and one permanence.

Now as no increase or decrease in the number of variations of signs can be produced by any of the functions changing signs, except the first, and as that changes as many times as it has real roots, therefore the changes in the number of variations of signs show the number of real roots comprised between h and h'.

If h and h' are taken at once at the widest limits of possibility, from — infinity to + infinity, the number of variations of signs will indicate the number of real roots; — and this number, taken from the degree of the equation, will give the number of the imaginary roots.

(Art. 187.) The foregoing is a full theoretical demonstration of the theorem; but the subject itself being a little *abstruse*, some minds may require the following practical elucidation.

Form an equation with the four assumed roots, 1, 3, 4, 6

The equation will be

$$(x-1)(x-3)(x-4)(x-6)=0;$$

or $X = x^4-14x^3+67x^2-126x+72=0$ *Roots* 1, 3, 4, 6.

$X = 4x^3-42x^2+134x-126 = 0$ . . *Roots* 2, 3.3, 5 nearly.

$R = 13x^2-91x+153$ . . . . *Roots* 2.8, 4.1 nearly.

$R' = 72x-252$ . . . . *Root* 3.5

$R'' = +$

Let the pupil observe these functions, and their roots, and see that they correspond with theory. The least root of X is less than the least root of X′. (Art. 183.) The roots of any function are intermediate between the roots of the adjacent functions. This corresponds with (Prop. 2.); for if three consecutive functions have the same sign as —, —, —, or +, +, +, the middle one cannot change first and correspond to (Prop. 2.); but signs change only by the *increasing* quantity passing a root, and it must pass a root of one of the extreme functions first; therefore the roots of X′ must be intermediate in value between the roots of X and R; and the roots of R intermediate in value between the roots of X′ and R′; and so on. But the roots of X′ are within narrower limits than the roots of X (Art. 183.); therefore the roots of all the functions are within the limits of the roots of X.

We will now trace all the changes of signs in passing all the roots of all the functions.

We will first suppose x or $h = 0$; which is less than any root; then as we increase h above any root, we must change the sign of that function, *and that sign only*

We represent these changes thus:

		X	X′	R	R′	R″	
*When	$x=0$	+	—	+	—	+	....4 variations
"	$x=1.1$	—*	—	+	—	+	....3 "
"	$x=2.1$	—	+*	+	—	+	....3 "
"	$x=2.9$	—	+	—*	—	+	....3 "
"	$x=3.1$	+*	+	—	—	+	....2 "
"	$x=3.4$	+	—*	—	—	+	....2 "
"	$x=3.7$	+	—	—	+*	+	....2 "
"	$x=4.1$	—*	—	—	+	+	....1 "
"	$x=4.11$	—	—	+*	+	+	....1 "
"	$x=5.1$	—	+*	+	+	+	....1 "
"	$x=6.1$	+*	+	+	+	+	....0 "

We commenced with 4 variations of signs, and end with 0 variations, after we have passed all the roots; therefore the real roots, in the primitive equation, must be $4-0=4$.

By this it can be clearly seen, by inspection, that the changes of sign in all the functions, except the first, produce *no* change in the number of variations.

In making use of this theorem we do not go through the intermediate steps, unless we wish to learn the locality of the roots as well as their number. We may discover their number by substituting a number for x *less* than any root, and then one greater; the difference of the variations of signs will be equal to the number of *real* roots.

If we take $-\infty$ and $+\infty$, the sign of any whole function will be the same as that of its first term.

---

* In making this table, we did not really substitute the numbers assumed for x, as we previously determined the roots; and as passing any root changes the sign in that *function*, we write a star against that sign which has just changed.

CHAPTER V.

APPLICATION OF STURM'S THEOREM.

(Art. 188.) In preparing the functions, remember that we are at liberty to suppress positive numeral factors.

EXAMPLES.

1. How many real roots has the equation $x^3+9x=6$?

Here $$X=x^3+9x-6$$ $$X'=x^2+3$$ $$R=-x+1$$ $$R'=-$$

Now for x substitute $-\infty$ or -100000, and we see at once that

X	X′	R	R′	
—	+	+	—	2 variations.

Again, for x put $+\infty$ or $+100000$, and the resulting signs must be

X	X′	R	R′	
+	+	—	—	1 variation.

Hence the above equation has but one *real* root; and, of course, two *imaginary* roots.

To find a near locality of this root, suppose $x=0$, and the signs will be

	X	X′	R	R′	
	—	+	±	—	2 variations.
$x=1$	+	+	±	—	1 variation.

Hence the real root is between 0 and 1

Now as we have found x, in the equation $x^3+9x-6=0$, to be less than 1, x^3 may be disregarded, and $9x-6=0$, will give us the first approximate value of x; that is, $x=.6$, nearly.

2. How many real roots has the equation $x^4-3x^2-4=0$?

$$X=x^4-3x^2-4$$ $$X'=4x^3-6x$$ $$R=+25$$

If	$x=-\infty$	+	—	+	2 variations.
	$x=+\infty$	+	+	+	0 variation.

Hence there must be 2 real roots, and 2 imaginary roots.

3. How many real roots has the equation $x^6-4x^3-621=0$? (See Art. 103.)

$$X=x^6-4x^3-621$$
$$X'=x^5-2x^2$$
$$R=+625$$

When $x=-\infty$ $+$ $-$ $+$ 2 variations.

When $x=+\infty$ $+$ $+$ $+$ 0 variation.

Hence there are two real roots and four imaginary roots.

4. How many real roots has the equation $x^3-15x+21=0$?
Ans. 3.

5. How many real roots are contained in the equation
$x^3-5x^2+8x-1=0$? *Ans.* 1.

6. How many real roots are contained in the equation
$2x^4-13x^2+10x-19=0$? *Ans.* 2.

7. Find the number and situation of the roots of the equation
$x^3+11x^2-102x+181=0$.

$$X=x^3+11x^2-102x+181$$
$$X'=3x^2+22x-102$$
$$R=122x-393$$
$$R=+$$

Putting $x=-\infty$ $-$ $+$ $-$ $+$ 3 variations.

$x=+\infty$ $+$ $+$ $+$ $+$ 0 variation.

Hence all the roots are real.

To obtain the locality of these roots there are several principles to guide us, there is (Art. 180), but the real limits are much narrower than that article would indicate, unless all the coefficients after the first are *minus*, and equal to the greatest.

A *practised eye* will decide nearly the value of a *positive* root by inspection; but by (Art. 183.) we learn that the root of R, or $122x-393=0$, must give a value to x intermediate between the roots of the primitive equation.

From this we should conclude at once that there must be a root between 3 and 4.

Substituting 3 for x, in the above functions, we have

$+ \quad - \quad - \quad +$ 2 variations.

$x=4 \quad + \quad + \quad + \quad +$ 0 variation.

Hence there are two roots between 3 and 4.

As the sum of the roots must be -11, and the two positive roots are more than 6, there must be a root *near* -17.

As there are *two* roots between 3 and 4, we will transform the equation, (Art. 175), into another, whose roots shall be less by 3; or put $x=3+y$. Then we shall have

$$X = y^3+20y^2-9y+1=0$$
$$X' = 3y^2+40y-9$$
$$R = 122y-27$$
$$R' = +$$

The value of y, in this transformed equation, must be near the value of y in the equation $122y=27$, (Art. 183.); that is, y is between .2 and .3

$y=.2$ gives $+ \quad - \quad - \quad +$ 2 variations.

$y=.3$ gives $+ \quad + \quad + \quad +$ 0 variation.

Hence there are two values of y between .2 and .3; and, of course, two values of x between 3.2 and 3.3.

We may now transform this last equation into another whose roots shall be .2 less, and further approximate to the true values of x, in the original equation.

Having thus explained the foregoing principles, and, in our view, been sufficiently elaborate in theory, we shall now apply it to the solution of equations, commencing with

NEWTON'S METHOD OF APPROXIMATION.

(Art. 189.) We have seen, in (Art. 175.), that if we have any equation involving x, and put $x=a+y$, and with this value transform the equation into another involving y, the equation will be

$$X+X'y+\frac{X''}{2}y^2+\frac{X'''}{2.3}y^3 \cdot \cdot \cdot \cdot \cdot y^m=0.$$

If a is the real value of x, then $y=0$, and $X=0$.

If a is a very near value to x, and consequently y very small, the terms containing y^2, y^3, and all the higher powers of y, become very small, and may be neglected in finding the *approximate* value of y.

Neglecting these terms, we have

$$X+X'y=0,$$

$$\text{Or} \quad \ldots\ldots \quad y=-\frac{X}{X'} \ldots\ldots\ldots\ldots (1)$$

In the equation $x=a+y$, if a is less than x, y must be positive; and if y is positive in the last equation, X and X′ must have opposite signs, corresponding to (Art. 184.).

Following formula (1), we have an approximate value of y; and, of course, of x. The value of x, thus corrected, again call a, and find a correction as before; and thus approximate to any required degree of exactness.

EXAMPLES.

1. Given $3x^5+4x^3-5x-140=0$, to find one of the approximate values of x.

By *trial* we find that x must be a little more than 2.

Therefore, put $x=2+y$.

$X= 3(2)^5+ 4(2)^3-5(2)-140 \ldots \text{or} \ldots X=-22$

$X'=15(2)^4+12(2)^2-5 \ldots\ldots\ldots \text{or} \ldots X'= 283.$

Whence $y=-\frac{X}{X'}=\frac{22}{283}=0.07$ nearly.

For a second operation, we have

$$x=2.07+y.$$

$X= 3(2.07)^5+ 4(2.07)^3-5(2.07)-140 \ldots \text{By log. } X=-0.854$

$X'=15(2.07)^4+12(2.07)^2-5 \ldots\ldots \text{By log. } X'= 321\ 82$

Hence the second value, or $y=\frac{8.54}{3218.2}=0.00265+$

And $\ldots\ldots\ldots\ldots\ldots\ldots$ $x=2.07265+$

2. Given $x^3+2x^2-23x=70$, to find an approximate value of x. *Ans.* 5.1345+.

3. Given $x^4-3x^2+75x=10000$, to find an approximate value of x. *Ans.* $9.886+$

4. Given $3x^4-35x^3-11x^2-14x+30=0$, to find an approximate value of x. *Ans.* $11.998+$.

5. Given $5x^3-3x^2-2x=1560$, to find an approximate value of x *Ans.* $7.00867+$.

CHAPTER VI.

HORNER'S METHOD OF APPROXIMATION.

(Art. 190.) In the year 1819, Mr. W. G. Horner, of Bath, England, published to the world the most elegant and concise method of approximating to roots of any then known.

The parallel between Newton's and Horner's method, is this; both methods commence by finding, by *trial*, a near value to a root.

In using Horner's method, care must be taken that the number, found by trial, be less than the real root. Following Newton's method we need not be particular in this respect.

In both methods we transform the original equation involving x, into another involving y, by putting $x=r+y$, as in (Art. 175), r being a rough approximate value of x, found by trial.

The transformed equation enables us to find an approximate value of y, (Art. 189.).

Newton's method puts this approximate value of y to r, and uses their algebraic sum as r was used in the first place; again and again transforming the *same* equation, after each successive correction of r.

Horner's method transforms the *transformed* equation into another whose roots are less by the approximate value of y; and again transforms that equation into another whose roots are less, and so on, as far as desired.

By continuing similar notation through the several transformations we may have

$$x = r + y$$
$$y = s + z$$
$$z = t + z'$$
$$z' = u + z''$$
&c. &c.

Hence $x = r + s + t$, &c.; r, s, t, &c., being successive figures of the root. Thus if a root be 325, $r = 300$, $s = 20$, and $t = 5$.

On the principle of successive transformations is founded the following

RULE *for approximating to the true value of a real root of an equation.*

1st. *Find by Sturm's Theorem, or otherwise, the value of the first one or two figures of the root, which designate by* r.

2d. *Transform the equation* (Art. 175), *into another whose roots shall be less by* r.

3d. *With the absolute term of this transformed equation for a dividend, and the coefficient of* y *for a divisor, find the next figure of the root.*

4th. *Transform the last equation into another, whose roots shall be less, by the value of the last figure determined; and so proceed until the whole root is determined, or sufficiently approximated to, if incommensurable.*

NOTE 1. In any transformed equation, X is a general symbol to represent the absolute term, and X′ represents the coefficient of the first power of the unknown quantity. If X and X′ become of the same sign, the last root figure is not the true one, and must be diminished.

NOTE 2. To find negative roots, change the sign of every alternate term, (Art. 178.): find the positive roots of that equation, and change their signs.

(Art. 191.) We shall apply this principle, at first, to the solution of equations of the second degree; and for such equations as have large coefficients and incommensurable roots, it will furnish by far the best *practical rule.*

EXAMPLES.

1. Find an approximate root of the equation

$$x^2+x-60=0.$$

We readily perceive that x must be more than 7, and less than 8, therefore $r=7$.

Now transform this equation into another whose roots shall be less by 7.

Operate as in (Art. 175.), synthetic division

$$\begin{array}{llll}
 & & & r \\
1 & 1 & -60 & (7 \\
 & \underline{7} & \underline{56} & \\
 & 8 & -4 & \\
 & \underline{7} & & \\
 & 15 & & \\
\hline
\end{array}$$

$$\text{Trans., eq.}\quad y^2 + 15y \qquad -4=0$$

Here we find that y cannot be far from $\frac{4}{15}$, or between .2 and .3; therefore transform the last equation into another whose roots shall be .2 less; thus,

$$\begin{array}{llll}
 & & & s \\
1 & 15 & -4 & (.2 \\
 & \underline{0.2} & \underline{3.04} & \\
 & 15.2 & -0.96 & \\
 & \underline{2} & & \\
 & 15.4 & &
\end{array}$$

The second transformed equation, therefore, is

$$z^2+15.4z-0.96=0$$

To obtain an approximate value of z, we have $\dfrac{.96}{15.4}$ or 0.06.

In being thus formal, we spread the work over too large a space, and must inevitably become tedious. To avoid these difficulties, we must make a few practical modifications.

1st. We will consider the absolute term as constituting the second member of the equation; and, in place of taking the algebraic sum of it, and the number placed under it, we will take their difference.

2d. We will *not* write out the transformed equations; that is, not attach the letters to the coefficients; we can then unite the whole in one operation.

3d. Consider the root a quotient; the absolute term a dividend, and, corresponding with these terms, we must have divisors.

In the example under consideration, 8 is the *first* divisor; 15 is the *first trial* divisor; 15.2 is the second divisor, and 15.4 is the second trial divisor; 15.46 is the third divisor, &c.

Let us now generalize the operation. The equation may be represented by

$$x^2+ax=n$$

Transform this into another whose roots shall be less by r; that equation into another whose roots shall be less by s, &c., &c.

SYNTHETIC DIVISION.

$$\begin{array}{lclll}
 & 1 & a & n & (\, r+s \\
 & & r & ar+r^2 & \\ \cline{3-4}
\text{1st divisor, } \ldots\ldots & & a+r & n' & \\
 & & r & (a+2r+s)s & \\ \cline{3-4}
\text{1st trial divisor, } \ldots & & a+2r & n'' & \\
 & & 0+s & \&c. & \\ \cline{3-3}
\text{2d divisor, } \ldots\ldots & & a+2r+s & & \\
 & & s & & \\ \cline{3-3}
\text{2d trial divisor, } \ldots & & a+2r+2s & & \\
 & & \&c. & &
\end{array}$$

In the above we have represented the difference between n and $(ar+r^2)$ by n', &c. As n', n'', n''', &c., with their corresponding trial divisors, will give s, t, u, &c., the following formula will represent the complete divisors for the solution of all equations in the form of

$$x^2\pm ax=\pm n$$

1st divisor, ...	$\pm a+r$
add	$r+s$
2d divisor, . .	$\pm a+2r+s$
add . .	$s+t$
3d divisor, ...	$\pm a+2r+2s+t$
add	$t+u$
4th divisor, . . .	$\pm a+2r+2s+2t+u$
&c	&c. &c.

Equations which have expressed coefficients of the highest power, as

$$cx^2\pm ax=n,$$

the formulas will be:

1st divisor, ...	$\pm a+cr$
add	$cr+cs$
2d divisor, ...	$\pm a+2cr+cs$
add	$cs+ct$
3d divisor, ...	$\pm a+2cr+2cs+ct$
&c.	&c.

To obtain *trial* divisors we would *add* cr only, in place of $(cr+cs)$, &c.

We will now resume our equation for a more concise solution.

1. $x^2+x=60$

		n	$r\ s\ t\ u$
	1	60	(7.262
	7	56	
1st divisor, . . .	8	4	
add . .	7.2	304	
2d divisor, . .	15.2	96	
add . .	26	9276	
3d divisor, . .	15.46	324	
add . .	62	31044	
4th divisor, . .	15.522	1356	

We can now divide as in simple division, and annex the quotient figures to the root, thus:

$$\begin{array}{r}
15522)1356\ \ (08734 \\
124176 \\ \hline
11424 \\
10865 \\ \hline
559 \\
465 \\ \hline
94
\end{array}$$

Hence, ... $x=7.2620873+$.

2. Find x, from the equation $x^2-700x=59829$.

On trial, we find x cannot exceed 800; therefore, $r=700$.

$$\begin{array}{llrr}
 & & & n \quad rst \\
a+r & -700+700= & 0 & 59829(777 \\
\quad r+s & 700+\ 70= & 770 & 00)000 \\ \hline
a+2r+s & \ldots\ldots & =770 & 770)59829=n' \\
\quad s+t=70+7 & & 77 & 5390 \\ \hline
a+2r+2s+t & & =847 & 847)5929=n'' \\
 & & & 5929 \\ \hline
\end{array}$$

Hence, ... $x=777$.

3 Find x from the equation $x^2-1283x=16848$.

By trial, we find that x must be more than 1000, and less than 2000; therefore, $r=1000$.

$$\begin{array}{lrr}
 & & n \quad rstu \\
a+r= & -283 & 16848(1296 \\
\quad r+s= & 1200 & -283 \\ \hline
a+2r+s= & 917 & 917)2998 \\
\quad s+t & 290 & 1834 \\ \hline
a+2r+2s+t= & 1207 & 1207)11644 \\
\quad t+u=96 & & 10863 \\ \hline
 & 1303 & 1303)\ \ 7818 \\
 & & 7818 \\ \hline
\end{array}$$

Hence, ... $x=1296$.

4. Given $x^2-5x=8366$, to find x.

By trial, we find x must be more than 90, and less than 100

$$
\begin{array}{lrl}
\text{Therefore,}\quad a+r & =85\;) & 8366\;(\;94=x \\
r+s\,.\,. & 94\;\; & 765 \\
\hline
a+2r+s & =179\;) & \;\;716 \\
 & & \;\;\underline{716}
\end{array}
$$

5. Find x, from the equation $x^2-375x+1904=0$.

Here the first figure of the root is 5.

$$
\begin{array}{lrr}
 & 5 & -1904(5.1480052207 \\
 & \underline{-375} & \underline{-1850} \\
\text{1st divisor,} & -370. & -5400 \\
 & \underline{5.1} & \underline{-3649} \\
\text{2d divisor,} & -364.9 & -175100 \\
 & \underline{.14} & \underline{-145904} \\
\text{3d divisor,} & -364.76 & -2919600 \\
 & \underline{48} & \underline{-2917696} \\
\text{4th divisor,} & -364.712 & -190400 \\
 & \underline{8} & \underline{-1823519} \\
\text{5th divisor,} & -364.7039 & -80491 \\
 & & \underline{-72941} \\
 & & -7550 \\
 & & \underline{-7294} \\
 & & -256 \\
 & & \underline{-255} \\
 & & -1
\end{array}
$$

6. Given $x^2+7x-1194=0$, to find x. *Ans.* 31.2311099.

7. Given $x^2-21x=214591760730$, to find x. *Ans.* 463251.

It might be difficult for the pupil to decide the value of r, as applied to the last example, without a word of explanation: x must be *more than the square root* of the absolute term, that is, more than 400000; then try 500000, which will be found too great.

(Art. 192.) When the coefficient of the highest power is not unity, we may (if we prefer it to using the last formulas for divisors), transform the equation into another, (Art. 166.), which shall have unity for the coefficient of the first term, and all the other coefficients whole numbers.

8. Given $7x^2-3x=375$.

Put $x=\frac{y}{7}$ and we shall have $y^2-3y=2625$.

One root of this equation is found to be 52.7567068+, one-seventh of which is 7.536672+; the approximate root of the original equation.

9. Given $7x^2-83x+187=0$, to find one value of x.

Ans. 3.024492664

10. Given $x^2-\frac{3}{11}x=8$, to find one value of x.

Ans. 2.96807600231

11. Given $4x^2+\frac{7}{9}x=\frac{1}{5}$, to find one value of x.

Ans. *Ans.* .14660+

12. Given $\frac{3}{4}x^2+\frac{3}{5}x=\frac{7}{11}$, to find one value of x.

Ans. .6042334631

13. Given $115-3x^2-7x=0$, to find one value of x.

Ans. 5.13368606

(Art. 193.) We now apply the same principle of transformation to the solution of equations of the third degree.

EXAMPLES.

1. Find one root of the equation

$$x^3-x^2+70x-300=0.$$

We find, by *trial*, that one root must be between 3 and 4.

	1st Coefficient.	2d Coefficient.	3d Coefficient.	4th Coefficient.	
					r
1st Transformation.	1	—1	70	—300	(3.
		3	6	228	
		2	*76	— 72=X	
		3	15	. . . .	
		5	91=X′	. . . .	$y=\frac{72}{91}=0.7$
		3	. .	. . . .	
		8	. .	. . . .	
					s
2d Transformation.	1	8	91	— 72	(0.7
		0.7	6.09	67.963	
		8.7	*97.09	— 4.037=X	
		7	6.58		
		9.4	103.67=X′		
		7			
		10.1			
					t
	1	10.1	103.67	— 4.037	(0.03
		.03	.3039	— 3.119217	
		10.13	*103.9739	— 0.917783=X	
		3	.3048		
		10.16	104.2787=X′		
		3			
		10.19			
					u
	1	10.19	104.2787	0.917783	(0.008
		.008	.081584	0.834882272	
		10.198	*104.360284	.082900728	
		8	81648		
		10.206	104.441932		
		8			
		10.214			

The terms here marked X are trial divisors; we have prefixed stars to the numbers that we may call complete divisors.

We rest here with the equation

$$(z'')^3+10.214(z'')^2+104.4419z''-0.0829=0.$$

The value of z'' is so small that we may neglect all its powers, except the first, and obtain several figures by division, thus:

$$\begin{array}{rl}
104\ )\ 829 & (\ 797 \\
\underline{728} & \\
101 & \\
\underline{936} & \\
74 &
\end{array}$$

Hence, one approximate value of x is . . . . $\overset{r}{3}.\overset{s}{7}\overset{t}{3}\overset{u}{8}797+$.

(Art. 193.) We may make the same remarks here as in (Art. 191.), and, as in that article, generalize the operation.

Let $x^3+Ax^2+Bx=N$ represent any equation of the third degree, and transform it into another whose roots shall be r less; thus,

$$\begin{array}{llllll}
1 & A & B & = & N & (\ r \\
 & \underline{r} & \underline{(r+A)r} & & \underline{r^3+Ar^2+Br} & \\
 & r+A & {}^*(r+A)r+B & & N' & \\
 & \underline{r} & & & & \\
 & 2r+A & \underline{(2r+A)r} & & & \\
 & \underline{r} & 3r^2+2Ar+B & & & \\
 & 3r+A & & & &
\end{array}$$

The transformed equation is

$$y^3+(3r+A)y^2+(3r^2+2Ar+B)y=N'.$$

If we put $(3r+A)=A'$, $(3r^2+2Ar+B)=B'$,

and . . . . $N-r^3-Ar^2-Br=N'$,

we shall have . . . $y^3+A'y^2+B'y=N'$, an equation similar to the primitive equation.

If we transform this equation into another whose roots shall be less by s, we shall have

$$z^3+(3s+A')z^2+(3s^2+2A's+B')z=N'',$$

Or, . . $z^3+A''z^2+B''z=N''$; an equation also similar to the first equation. And thus we may go on forming equation after equation, similar to the first, whose roots are less and less.

The quantities N', N'', &c., are the same as X in our previous notation, and the quantities B', B'', &c., are the same as the general symbol, X′; but we have adopted this last method of notation to preserve similarity.

Observe, that as $(r+A)r+B$ is the first complete divisor, N is the number considered as a dividend and r the quotient; and therefore . . . $r=\dfrac{N}{(r+A)r+B}$, nearly.

The next equation gives us

$$s=\frac{N'}{(s+A')s+B'}=\frac{N'}{s(s+3r+A)+B'},\text{ nearly.}$$

And the next, $t=\dfrac{N''}{(t+A'')t+B''}=\dfrac{N''}{t[t+3(r+s)+A]+B''}$ very nearly.

$$u=\frac{N'''}{(u+A''')u+B'''}=\frac{N'''}{u[u+3(r+s+t)+A]+B'''}$$

$$\&c. \quad = \quad \&c. \quad \&c.$$

The denominators of these fractions are considered complete divisors, and the quantities, B', B'', B''', are considered *trial* divisors. The further we advance in the operation the nearer will the *trial* and true divisors agree.

Before the operation is considered as commenced, we must find the first figure of the root (r) by *trial*. Then the operator can experience no serious difficulty, provided he has in his mind a clear and distinct method of forming the divisors; and these may be found by the following

RULE. 1st. *Write the number represented by* B, *and directly under it, write the value of* r(r+A); *the algebraic sum of these two numbers is the first complete divisor.*

2d. *Directly under the first divisor write the value of* r^2, *and the sum of the last three numbers is* B', *or the first trial divisor.*

3d. *Find by trial, as in simple division, how often* B′ *is contained in* N', *calling the first figure* s, *(making some allowance for the augmentation of* B′*), and* s *will be a portion of the root under trial.*

4th. *Take the value of the expression* $(3r+s+A)s$ *and add it to the first trial divisor; the sum is the second divisor* (*if we have really the true value of* s).

IN GENERAL TERMS;

Under any complete divisor, write the square of the last figure of the root; add together the three last columns, and their sum is the next trial divisor. With this trial divisor decide the next figure of the root.

Take the algebraic sum, of three times the root previously found, the present figure under trial, and the coefficient $\mathcal{A}$, and multiply this sum by the figure under trial, and this product, added to the last trial divisor, gives the next complete divisor.

EXAMPLES.

1. Given $x^3+2x^2+3x=13089030$, to find one value of x.

By trial, we soon find that x must be more than 200 and less than 300; therefore we have

$$r=200, \quad \mathcal{A}=2, \quad B=3.$$

By the rule,

			N	rs
B	3	40403)	13089030	(235
$r(r+\mathcal{A})$	40400		80806	
1st divisor	40403	139763)	500843	$=N'$
r^2	40000		419289	
1st trial divisor .. B'	$=120803$	163108)	815540	$=N$
$(3r+s+\mathcal{A})s$ ...	18960		815540	
2d divisor	139763			
s^2	900			
2d trial divisor .. B''	$=159623$			
$(3r+3s+t+\mathcal{A})t$ ..	3485			
3d divisor	163108			

Hence, ... $x=235$.

2. Given $x^3+173x=14760638046$, to find one value of x

Here $\mathcal{A}=0$, $B=173$, and we find, by trial, that x must be more than 2000, and less than 3000; therefore $r=2000$

$$
\begin{array}{rl}
B\ldots\ldots\ldots & 173\\
\begin{array}{r} r(r+\mathcal{A})\ldots\ldots\\ \text{1st divisor}\ldots\ldots\ldots\\ r^2\ldots\ldots\ldots \end{array} & \left.\begin{array}{r} 4000000\\ \hline 4000173\\ 4000000 \end{array}\right\}\\
\cline{2-2}
B'\ldots\ldots\ldots & 12000173\\
\begin{array}{r} (3r+s+\mathcal{A})s\ldots\ldots\\ \text{2d divisor}\ldots\ldots\ldots\\ s^2\ldots\ldots\ldots \end{array} & \left.\begin{array}{r} 2560000\\ \hline 14560173\\ 160000 \end{array}\right\}\\
\cline{2-2}
B''\ldots\ldots\ldots & 17280173\\
\begin{array}{r} (3r+3s+t+\mathcal{A})t\ldots\\ \text{3d divisor}\ldots\ldots\ldots\\ \; \end{array} & \left.\begin{array}{r} 362500\\ \hline 17642673\\ 2500 \end{array}\right\}\\
\cline{2-2}
B'''\ldots\ldots\ldots & 18007673\\
(3R+u+\mathcal{A})u\ldots\ldots & 22059\\
\cline{2-2}
\text{4th divisor}\ldots\ldots\ldots & 18029732
\end{array}
$$

$$
\begin{array}{rrl}
4000173\;) & 14760638046 & (\;2453=x\\
 & 8000346 & \\
\cline{2-2}
14560173\;) & 67602920 & =N'\\
 & 58240692 & \\
\cline{2-2}
17642673\;) & 93622284 & =N''\\
 & 88213365 & \\
\cline{2-2}
18029732\;) & 54089196 & =N'''\\
 & 54089196 & \\
\cline{2-2}
\end{array}
$$

3. Given $x^3+2x^2-23x=70$, to find an approximate value of x. *Ans.* $x=5.134578+$.

4. Given $x^3-17x^2+42x=185$, to find an approximate value of x *Ans.* $x=15.02407$.

(Art. 194.) When the coefficient of the highest power is not unity, we may transform the equation into another, (Art. 166.), in which the coefficient of the first term is unity, and all the other coefficients whole numbers; but it is more direct and concise to modify the *rule* to suit the case.

If the coefficient of the first power is c, the first divisor will be $(cr+\mathcal{A})r+B$, in place of $(r+\mathcal{A})r+B$.

In place of $(3r+s+\mathcal{A})s$, to correct the first trial divisor, we must have $(3cr+cs+\mathcal{A})s$; and, in general, in place of using 3 times the root already found, we must use $3c$ times the root; and, in place of the square of any figure, as r^2, s^2, &c., we must use cr^2, cs^2, &c.

EXAMPLES.

5. Find one root of the equation, $3x^3+2x^2+4x=75$.

By trial, we find that x must be more than 2, and less than 3; therefore $r=2$, $c=3$, $\mathcal{A}=2$, $B=4$.

		N $\quad r\,s\,t\,u$
B	4.	75 (2.577
$(cr+\mathcal{A})r$	16.	40
1st divisor	20.	$35=N'$
cr^2	12.	29375
B'	48.	$5625=N''$
$(3cr+cs+\mathcal{A})s$ ..	10.75	5038579
2d divisor	58.75	$.586421=N'''$
cs^2	75	.517301099
B''	70.25	69119901
*$(3cR+ct+\mathcal{A})t$.	1.7297	
3d divisor	71.9797	Continue, by simple division, thus
ct^2	147	739) 6911 (935
B'''	73.7241	6651
$(3cR+cu+\mathcal{A})u$.	176057	260
4th divisor	73.900157	221

Hence, ... $x=2.577935+$.

---

* R is a symbol to represent the entire root, as far as determined.

6. Find one root of the equation, $5x^3-6x^2+3x=-85$.
Ans. $x=-2.16399-$.

7. Find one root of the equation, $12x^3+x^2-5x=330$.
Ans. $x=3.036475+$

8 Find one root of the equation, $5x^3+9x^2-7x=2200$.
Ans. $x=7.1073536+$.

9. Find one root of the equation, $5x^3-3x^2-2x=1560$.
Ans. $x=7.0086719+$.

(Art. 195.) This principle of resolving cubic equations may be applied to the extraction of the cube root of numbers, and indeed gives one of the best practical rules now known.

For instance, we may require the cube root of 100. This gives rise to the equation

$$x^3+Ax^2+Bx=100\,;$$

in which $A=0$, and $B=0$, and the value of x is the root sought.

As A and B are each equal to zero, the rule under (Art. 193.) may be thus modified.

1st. Keeping the symbols as in (Art. 193.), and finding r by *trial*, r^2 *will be the first divisor, and* $3r^2$ *is* B′, *or the first* TRIAL *divisor.*

2d. *By means of the dividend (so called), and the first trial divisor, we decide* s *the next figure of the root.*

3d. *Then* $(3r+s)s$; *that is, three times the portion of the root already found, with the figure under trial annexed, and the sum multiplied by the figure under trial, will give a sum, which, if written two places to the right, under the last trial divisor, and added, will give the next complete divisor.*

4th. After we have made use of any complete divisor, write the square of the last quotient figure under it; the sum of the three preceding columns is the next *trial* divisor; which use, and render complete, as above directed, and so continue as far as necessary.*

---

* In case of approximate roots after three or four divisors are found, we may find two or three more figures of the root, with accuracy, by simple division.

We may now resolve the equation

$$x^3=100, \qquad r=4.$$

1st divisor	16	$r\ stu$
$B'=3r^2$ ...	48	100 (4.6415889+
$(3r+s)s$ ...	756	64
2d divisor	5556	36
s^2	36	33336
B''	6348	2664000
$(3R+t)t$ ...	5536	2561344
3d divisor	640336	102656
t^2	16	64602721
B'''	645888	38053279
$(3R+u)u$ ...	13921	32308321
4th divisor	64602721	5744958
	1	5169331
	64616643	575627
	− 646166	516933
		58694
		58154

2. Extract the cube root of 673373097125.

1st divisor ..	64	N $\quad rstu$
B' ...	192	673373097125 (8765
$(3r+s)s$.	1729	512
2d divisor ..	20929	161373
	49	146503
B'' ...	22707	14870097
$(3R+t)t$.	15696	13718376
3d divisor ..	2286396	1151721125
	36	1151721125
B'''	2302128	
	131425	
	230344225	

NOTE.—To determine s, we have 192)1613(Some allowance made for the increase of 192.

3. Extract the cube root of 1352605460594688.

Ans. 110592.

4. Extract the cube root of 5382674. *Ans.* 175.25322796.

5 Extract the cube root of 15926.972504.

Ans. 25.16002549

6. Extract the cube root of 91632508641.

Ans. 4508.33859058

7. Extract the cube root of 483249. *Ans.* 78.4736142

(Art. 196.) The method of transforming an equation into another, whose roots shall be less by a given quantity, will resolve equations of any degree; and for all equations of higher degrees than the third, we had better use the original operation, as in (Art. 192.), and attempt no other modification than conceiving the absolute term to constitute the second member of the equation; and the *difference* of the numbers taken in the last column in place of their algebraic sum.

The following operation will sufficiently explain:

1. Find one value of x from the equation

$$x^4 - 3x^2 + 75x = 10000.$$

1st Transformation.

					r
1	±0	—3	75	=10000	(9
	9	81	702	6993	
	9	78	777	3007=N	
	9	162	2160		
	18	240	2937		
	9	243			
	27	483			
	9				
	36				

(Continued on the next page.)

2d Transformation.

				s
1	36	483.	2937.	=3007. (0.8
	.8	29.44	409.952	2677.5616
	36.8	512.44	3346.952	329.4384=N''
	8	30.08	434.016	
	37.6	542.52	3780.968	
	8	30.72		
	38.4	573.24	Take the coefficients to their nearest unit.	
	8			
	39.2			

				t
1	39	573	3781	= 329.4384 (0.08
	+	3	46	306.16
	39	576	3827	23.2784=N'''
	+	3	46	
	39	579	3873	

				u
1	39	579	3873	= 23.2784(0.00600+
			3	23.256
			3876	224
			3	
			3879	

Hence, ... $x=9.88600+$.

N. B. We went through the first and second transformations in full. Had we been exact, in the third, we should have added .08 to 39.2, and multiplied their sum, (39.28), by .08, giving 3.1424; we reserve 3. only to add to the next column. By a similar operation we obtain 46. to add to the next column.

EXAMPLES.

1. Given $x-x^2-x^3-x^4+500=0$, to find one value of x. *Ans.* 4.46041671

2. Given $x^4-5x^3+9x=2.8$, to find one value of x. *Ans.* .32971055072

3. Given $20x+11x^2+9x^3-x^4=4$, to find one value of x. *Ans.* .17968402502

4. Required the 5th root of 5000; or, in other terms, find one root of the equation $x^5=5000$. *Ans.* 5.49280+

5. Given $x^3=\left(\frac{8x}{x^2+1}\right)^2$ to find one value of x. *Ans.* 2.120003355

In following out general principles, we have, thus far, omitted some artifices which apply in particular cases, or depend on particular circumstances, which we will now set forth, as they will enlarge the views of all who pay attention to them.

The first circumstance to be observed is the sum of the numerical coefficients.

(Art. 197.) Transpose every term to the first member of the equation, and if the sum of the coefficients *is zero*, we are sure that *unity is one of the roots of the equation.*

For example, we are sure that the equation

$$x^4+7x^3-13x^2-79x+84=0,$$

has *unity* for one of its roots, because $1+7-13-79+84=0$. Assume $x=1$, and substitute 1 for x in the equation, and the equation will be verified. But assume x, a little greater or a little less than 1, and substitute, and the equation cannot be verified; therefore the following principle is established:

In any equation having numerical coefficients, if the sum of the coefficients is zero, $+1$ *is one of the roots of the equation.*

We can now divide the given equation by $(x-1)$ and thus depress it one degree.

For another example we will recall the 4th, on page 266, which is

$$x^3-13x+12=0.$$

Here $1-13+12=0$, whence $x=1$. The answer on page 266 is 3; that is another root. The third root must, therefore, be -4; the sum of all the roots must be 0, because the second term of the equation is zero. (Art. 157, second observation.)

We perceive at once, that *unity* is one root in each of the following equations:

1. $x^3-9x^2+24x-16=0.$
2. $x^3-6x^2+11x-6=0.$
3. $x^4-8x^3+14x^2+8x-15=0.$
4. $x^3-13x^2+39x-27=0.$

(Art. 198.) When the absolute term of an equation is numerically too large (whether it be plus or minus), to make the sum of the coefficients zero, we can diminish it, as well as other coefficients, by the following transformation.

For instance: In the equation

$$x^3-10x^2+11x+70=0,$$

the sum of the coefficient is obviously not equal to zero; 70 is too large, but we can diminish it by placing $x=2y$, or $x=ny$, and substituting this value of x in the equation, we have

$$n^3y^3-10n^2y^2+11ny+70=0.$$

Dividing by n^3 we obtain

$$y^3-\frac{10}{n}y^2+\frac{11}{n^2}y+\frac{70}{n^3}=0.$$

Now, we can assume n equal to 2, 3, 4, or any other number whatever.

Let $n=2$, and the preceding equation becomes

$$y^3-5y^2+\frac{11}{4}y+\frac{70}{8}=0. \qquad (1)$$

Let $n=-2$, $\quad y^3+5y^2+\frac{11}{4}y-\frac{35}{4}=0. \qquad (2)$

Let $n=5$, and $\quad y^3-2y^2+\frac{11}{25}y+\frac{14}{25}=0. \qquad (3)$

The sum of the coefficients in each of the equations (2) and (3), is zero. Therefore, y equals 1 in each of them.

But $\quad x=ny$, therefore $x=-2$, and $x=5$,
which are two of the three roots in the original equation.

But the sum of the three roots must be 10, by the general theory of equations; therefore, the third root must be 7,

Because, $\quad -2+5+7=10.$

If we take the first equation in (Art. 197), and depress it one degree, by dividing by $(x-1)$, we shall obtain the equation

$$x^3+8x^2-5x-84=0. \qquad (a)$$

Here it is obvious that the coefficients require depressing to make their sum zero.

Therefore, place $\quad x=nP$, and

$$P^3+\frac{8}{n}P^2-\frac{5}{n^2}P-\frac{84}{n^3}=0. \qquad (b)$$

Making $n=3$, this equation becomes

$$P^3+\frac{8}{3}P^2-\frac{5}{9}P-\frac{28}{9}=0.$$

Here $\frac{9}{9}+\frac{24}{9}-\frac{5}{9}-\frac{28}{9}=0$. Whence $P=1$, but $n=3$, and $x=nP=3$, another of the roots of the original equation.

We may now depress equation (a) one degree, by dividing by ($x-3$), and then find the other two roots.

Or, we may assume $n=-4$, and substitute this value for n in (b), then that equation becomes

$$P^3-2P^2-\tfrac{5}{16}P+\tfrac{21}{16}=0.$$

Here, because $\tfrac{16}{16}-\tfrac{32}{16}-\tfrac{5}{16}+\tfrac{21}{16}=0$. $P=1$, and $x=-4$, which is a third root of the original equation. $x=-7$ is the fourth root.

Thus we can find all the roots in the equation contained in the last article.

If it were our object to transform an equation so as to increase the coefficients in place of diminishing them, as in this article, we should place $x=\frac{P}{n}$, but this has already been shown in (Art. 166).

This method of solution may be considered a sequence to Newton's method of dividers as shown in (Art. 147).

(Art. 199.) An equation which has *one* for a root, can be changed into another, which will have *minus one* for a root, by changing its second, and every alternate sign. (Art. 178.)

Consequently, then, an equation which has *minus* one for one of its roots, can be changed into another, which will have *plus one* for one of its roots, by changing the sign of its second, and every alternate term.

Therefore, if *changing the sign of the second, and every alternate term in any equation will make the sum of the coefficients* ZERO, *one root of that equation must be* -1.

For example, the sum of the coefficients of the equation

$$4x^3-20x^2+11x+35=0, \text{ is not zero};$$

but change the signs as above directed, and we have the equation,

$$4x^3+20x^2+11x-35=0,$$

and the sum of the coefficients is now zero, and consequently *unity* is one root of this equation, and therefore -1 is a root of the first mentioned equation.

Hence, in seeking to solve any equation of this kind, if the sum of the coefficients is not zero, we may change the sign of the second, and each alternate term, and if the sum is then zero, -1 may be taken as one root of the equation; and then we may depress the equation one degree by dividing by $(x+1)$.

(Art. 200.) Another circumstance to be observed in relation to coefficients, is that of their *recurrence*, producing what is called

RECURRING EQUATIONS.

In a recurring equation the coefficients of those terms which are equally distant from the extreme terms, are numerically equal.

$$\text{Thus}\quad x^5-3x^4+7x^3+7x^2-3x+1=0, \qquad (1)$$
$$x^5+3x^4-7x^3+7x^2+3x+1=0, \qquad (2)$$
$$x^5+3x^4-7x^3+7x^2-3x+1=0, \qquad (3)$$
$$x^5-3x^4+7x^3-7x^2+3x-1=0, \qquad (4)$$

are all recurring equations of an odd degree.

If we substitute -1 for x, in each of the equations (1) and (2), they will be verified; and $+1$ put for x will verify (3) and (4). Therefore, any equation of an odd degree has either -1 or $+1$ for one of its roots, and the equation itself is reducible to an even degree, by dividing by $(x+1)$ or by $(x-1)$.

Minus one is a root, when the like coefficients have the same sign; and plus one is a root when the like coefficients have opposite signs.

(Art. 201.) *Recurring equations, of an even degree, in which the like coefficients have opposite signs, and whose middle term is wanting,* are divisible by (x^2-1), and therefore $+1$, and -1, are both roots of every such equation.

The following is an equation of this description:

$$x^6+8x^5+11x^4-11x^2-8x-1=0,$$

which can be put in the following form:

$$(x^6-1)+8x(x^4-1)+11x^2(x^2-1)=0.$$

and this is obviously divisible by (x^2-1). Whence we may place $x^2-1=0$, and $x=1$, or -1.

And thus the principle enunciated in this article is established. Dividing the equation by (x^2-1), will reduce it two degrees lower, and leave it an equation of an even degree.

(Art. 202.) *Every recurring equation of an even degree, above the second, can be reduced to an equation of half that degree, by the following artifice.*

We take the following equation for an example:

$$x^6-7x^5+8x^4-9x^3+8x^2-7x+1=0. \qquad (1)$$

Divide each term by the *square root* of the highest power of the unknown quantity. In this case divide by x^3, and we have

$$x^3-7x^2+8x-9+\frac{8}{x}-\frac{7}{x^2}+\frac{1}{x^3}=0. \qquad (2)$$

Which can be put in the following form:

$$\left(x^3+\frac{1}{x^3}\right)-7\left(x^2+\frac{1}{x^2}\right)+8\left(x+\frac{1}{x}\right)-9=0. \quad (3)$$

Now assume, $\left(x+\frac{1}{x}\right)=z.$ Then, $\left(x^2+\frac{1}{x^2}\right)=z^2-2.$

And $\left(x^3+\frac{1}{x^3}\right)=z^3-3z.$ Whence (3) becomes

$$(z^3-3z)-7(z^2-2)+8z-9=0, \qquad (4)$$

an equation of the third degree.

Thus (1), an equation of the 6th degree is reduced to (4), an equation of half that degree; and this process will reduce any other recurring equation of an even degree to another of half that degree.

(Art. 203.) A recurring equation of the 4th degree can be reduced to a quadratic by the following rule.

Divide by the coefficient of x^4, and transpose the term containing x^2. Add to each member $\pm 2x^2$. Use the plus sign when the signs *of the second and fourth term are alike, and use the minus sign when they are unlike. Also add to each member the square of half the term containing x. Extract the square root of each member, and the result is an equation of the second degree.*

EXAMPLES.

1. $ax^4+2bx^3+cx^2-2bx+a=0,$

$$x^4+\frac{2b}{a}x^2-\frac{2b}{a}x+1=-\frac{c}{a}x^2,$$

$$x^4+\frac{2b}{a}x^3-2x^2+\frac{b^2}{a^2}x^2-\frac{2b}{a}x+1=\left(-2+\frac{b^2}{a^2}-\frac{c}{a}\right)x^2$$

$$x^2+\frac{b}{a}x-1=\pm\left(\sqrt{-2+\frac{b^2}{a}-\frac{c}{a}}\right)x, \text{ a quadratic.}$$

2. $x^4+8x^3-5x^2+8x+1=0,$

$$\begin{array}{ll} & x^4+8x^3+2x^2+8x+1=7x^2 \\ \text{Add } (4x)^2 & \qquad\quad\ 16x^2 \qquad\quad =16x^2 \\ \hline & x^4+8x^3+18x^2+8x+1=23x^2 \end{array}$$

$$x^2+4x+1=\pm\sqrt{23}.x, \qquad \text{a quadratic.}$$

We are indebted to Professor Stevens, of Greenmount College, Indiana. for this article. It is original with him.

BINOMIAL EQUATIONS.

(Art. 204.) Binomial equations have but two terms, such as

$$x-1=0, \quad x^2-1=0, \quad x^3-1=0, \text{ \&c.}$$

Or, such as $x\pm a=0$, $x^n\pm a^n=0$, &c., &c.

Every binominal equation is also a recurring equation, for the coefficient certainly recurs.

A binomial equation has apparently but one root, but a full investigation will discover as many roots as there are units in the exponent of the unknown quantity, as is explained by an example in (Art. 163). We here give one more practical illustration.

The equation $x^5-1=0$, is a binomial equation of an odd degree, and it is also a recurring equation.

It is also divisible by $(x-1)$, having apparently but one root, $+1$. The division produces

$$x^4+x^3+x^2+x+1=0. \qquad (1)$$

a recurring equation of the 4th degree.

Dividing again by x^2, we obtain

$$x^2+x+1+\frac{1}{x}+\frac{1}{x^2}=0. \qquad (2)$$

Assuming $\left(x+\frac{1}{x}\right)=P$. Then $x^2+\frac{1}{x^2}=P^2-2$,

And (2) becomes $P^2+P-1=0$, a quadratic.

Again: Operating under (Art. 203), we obtain

$$\begin{array}{lrcl} & x^4+x^3+2x^2+x+1 & = & x^2 \\ \text{Add } \left(\frac{x}{2}\right)^2 & \frac{x^2}{4} & = & \frac{x^2}{4} \\ \hline & x^4+x^3+\frac{9x^2}{4}+x+1 & = & \frac{5}{4}x^2 \end{array}$$

Square root, $x^2+\frac{1}{2}x+1=\pm\frac{x}{2}\sqrt{5}$

Or $x^2+\frac{1}{2}(1\mp\sqrt{5})x=-1$, a quadratic.

Whence the five roots in $(x^5-1)=0$, or the five-fifths roots of unity are

$$\begin{aligned} x=1, \text{ or } x&=\tfrac{1}{4}(\sqrt{5}-1+\sqrt{-10-2\sqrt{5}}) \\ \text{or } x&=\tfrac{1}{4}(\sqrt{5}-1-\sqrt{-10-2\sqrt{5}}) \\ \text{or } x&=-\tfrac{1}{4}(\sqrt{5}+1-\sqrt{-10+2\sqrt{5}}) \\ \text{or } x&=-\tfrac{1}{4}(\sqrt{5}+1+\sqrt{-10+2\sqrt{5}}). \end{aligned}$$

EXAMPLES.

1. Find the roots of the equation

$$x^5-5x^3+5x^2-1=0.$$

Ans. Three of its roots, are each equal 1, and $\left(\frac{3}{2}\pm\frac{1}{2}\sqrt{5}\right)$

2. $x^4+5x^3+2x^2+5x+1=0.$

Ans. $x=\pm\sqrt{-1}$, or $\frac{1}{2}(-5\pm\sqrt{21})$.

3 $x^4-4x^3+4x-1=0.$

See (Art. 201). *Ans.* $x=\pm1$, or $(2\pm\sqrt{3})$.

4. $x^4-\frac{5}{2}x^3+2x^2-\frac{5}{2}x+1=0.$

Ans. $x=2$, or $\frac{1}{2}$, or $\pm\sqrt{-1}$.

5. $5x^4+8x^3+9x^2+8x+5=0.$

(Art. 203) reduces this to the following quadratic:

$$x^2+\tfrac{4}{5}x+1=\frac{x}{5}\sqrt{21}.$$

6. $4x^6-24x^5+57x^4-73x^3+57x^2-24x+4=0.$

Roots, $2,\ \tfrac{1}{2},\ 2,\ \dfrac{1+\sqrt{-3}}{2}$, and $\dfrac{1-\sqrt{-3}}{2}$.

7. $4x^4+3x^3-8x^2-3x+4=0.$

Ans. $x=\pm1$, or $\dfrac{-3\pm\sqrt{73}}{8}$.

8. $x^4+24x^3-2x^2-24x+1=0.$

Ans. $x=\pm1$, or $-12\pm\sqrt{145}$.

Some one, or more of the foregoing principles will apply to the solution of the following examples.

9. $x^4-2x^3-7x^2-8x+16=0.$

Ans. $x=1$, or 4, or $\dfrac{-3\pm\sqrt{-7}}{2}$.

10. $x^4+2x^3-3x^2-4x+4=0.$

Ans. $x=1$, 1, or -2, -2.

11. $x^4-2x^3-25x^2+26x+120=0.$

Ans. $x=3$ or 5, or -2, or -4.

12. $x^4-2x^3+2x^2-x=0.$

Ans. $x=0$, or 1, or $\dfrac{1\pm\sqrt{-3}}{2}$.

13. $x^4-4x^3+8x^2-32x=0.$

Ans. $x=0$, or 4, or $\pm\sqrt{-8}$.

14. $x^3+5x^2+3x-9=0.$

Ans. $x=1$, or -3, or -3.

15. $x^3+6x^2-7x-60=0.$

Ans. $x=3$, -4, or -5.

16. $x^3+8x^2+17x+10=0.$

Ans. $x=-1$, or -2, or -5.

17. $x^3-29x^2+198x-360=0.$

Ans. $x=3$, 6, or 20.

18. $4x^3-112x^2+109x-27=0.$

Ans. $x=\frac{1}{2}, \frac{1}{2}$, or 27.

19. $4x^3-152x^2+149x-37=0.$

Ans. $x=\frac{1}{2}, \frac{1}{8}$, or 37.

20. $4x^4+3x^3+8x^2-3x+4=0.$

Ans. $x=-a\pm\sqrt{1}+a^2$ and $2a=(\frac{3}{6}\pm\sqrt{-247})$.

21. $x^5-13x^4+67x^3-171x^2+216x-108=0.$

(See Art. 198.) *Ans.* 3, 3, 3, or 2, 2.

Remark.—This treatise has hitherto omitted one process of elimination, which Bourdon, and some others, have introduced, and which, some teachers regard as very scientific and interesting. We cannot quite agree with this class of teachers, for whatever is most simple, and most practical, we regard as most scientific, and this method is neither very simple, nor practical. Still, in a work like this, it is worthy of a passing investigation like the following.

GENERAL METHOD OF ELIMINATION AMONG EQUATIONS ABOVE THE FIRST DEGREE.

(Art. 205.) Suppose we have two equations, each containing x and y, like the following:

$$\left.\begin{array}{ll} x+y-6=0. & (1) \\ x^3+y^3-72=0. & (2) \end{array}\right\} \quad Ans. \quad \begin{array}{l} x=4, \text{ or } 2. \\ y=2, \text{ or } 4. \end{array}$$

To make the illustration clear, we give the values of x and y. Now by the theory of equations (Art. 156), if $x=4$, equation (1) is divisible by $(x-4)$, without a remainder.

That is, whatever the remainder *appears to be, it is in value zero.*

Dividing $(x+y-6)$ by $(x-4)$ the quotient is 1, and the apparent remainder is $(y-2)$, but this must be zero; therefore $y=2$, corresponding to $x=4$.

But $x=4$, corresponds to both the given equations; therefore equation (2) is divisible by $(x-4)$, and $(x-4)$ *is a common measure to the two equations* (1) *and* (2).

Now, therefore, if we take equations (1) and (2), and operate with them for common measure, as taught in (Art. 27), the last remainder must be zero; and in fact we shall show, *that each and every remainder, throughout the whole operation, must be zero.* And if we take care that only one letter shall appear in the last remainder—that is, operate until one letter only appears in the remainder—that remainder can be put equal to zero, and the desired elimination is effected.

OPERATION.

$$\begin{array}{lll}
x+y-6)x^3+y^3-72(x^2 & & \\
\quad \underline{x^3+x^2y-6x^2\qquad\quad} & (A) & \\
\quad -x^2y+6x^2+y^3-72 & (B) & \text{remains.}
\end{array}$$

Observe that the divisor is in value zero, and it being multiplied by x^2, the product (A) must be zero in value; and that product, subtracted from the dividend (zero), the remainder (B) must be zero.

Thus we might establish the fact, that each remainder (and in fact each product also), throughout the operation, must be zero.

Continuing the operation we obtain

$$\begin{array}{lll}
 & -x^2y+6x^2+y^3-72(y^2 & \\
 & \underline{\;xy^2+y^3-6y^2\qquad\qquad\qquad\qquad} & \\
 & -x^2y-xy^2+6x^2+6y^2-72. & \text{2d remainder.} \\
\text{Or,} & -xy(x+y)+6(x^2+y^2)-72 & =\text{2d remainder.}
\end{array}$$

But we learn by equation (1) that $(x+y)=6$, therefore we can divide this second remainder by 6, and we have

$$\begin{array}{lll}
 & x^2-xy+y^2-12(x+(6-2y) & \\
 & \underline{x^2+xy-6x\qquad\quad} & \\
 & 6x-2xy+y^2-12 & \text{3d remainder.} \\
\text{Or,} & (6-2y)x+y^2-12 & \text{3d remainder.} \\
 & \underline{(6-2y)x+(6-2y)(y-6)\qquad\quad} & \\
 & \qquad y^2-12-(6-2y)(y-6). & \text{4th remainder.}
\end{array}$$

This last remainder does not contain x, therefore that letter is completely eliminated, and the remainder put equal to zero will give the values of y.

N. B.—When this method is clearly analyzed, it will be found to be nothing more, than elimination by addition and subtraction.

Here we have two equations, (1) and (2), each equal to zero, and we multiply one of them, and subtract the product from the other. The remainder is an equation equal to zero, and if its terms have a common divisor, we can divide the equation by it. Hence we have no new principle, nothing to gain by keeping these forms in view. The common method gives us much more freedom of action.

We give one more example.

$$\begin{array}{ll} \text{Given,} & \left.\begin{array}{l}(x^2+y^2)(x-y)-1220=0 \\ x^2+y^2+x-y\ -\ 132=0\end{array}\right\} \text{To find } x \text{ and } y. \\ \text{And,} & \end{array}$$

Assume, $x^2+y^2=P$, and $x-y=Q$,

Then, $P+Q-132=0$,

And, $PQ-1220=0$ $(Q$,

$$PQ+Q^2-132Q,$$

$$\overline{-Q^2+132Q-1220=0}, \text{ a quadratic.}$$

We extract the following from Prof. Perkin's Algebra. Indeed, we have observed it in several other works, and it is given as an appropriate example to illustrate this method of elimination.

We give it to show the utter inutility of this mode of operation, in a practical point of view.

Given, $x^2+xy+y^2-1=0$, (1)

And, $x^3+y^3=0$, (2)

to find a single equation in terms of y.

FIRST OPERATION.

$$x^2+xy+y^2-1)\ x^3+y^3\ (x-y$$

$$x^3+x^2y+xy^2-x$$

$$\overline{-x^2y-(y^2+1)x+y^3}$$

$$-x^2y-\ y^2x\ +y-y^3$$

$$\overline{x+2y^3-y}=\text{first remainder.}$$

SECOND OPERATION.

$$x+(2y^3-y)\ )x^2+xy+y^2-1(x-(2y^3-2y)$$

$$\underline{x^2+(2y^3-y)x}$$

$$-(2y^3-2y)x+y^2-1$$

$$\underline{-(2y^3-2y)x-4y^6+6y^4-2y^2}$$

$$4y^6-6y^4+3y^2-1=$$

second remainder.

Whence, $4y^6-6y^4+3y^2-1=0$, is the equation sought.

As the sum of these coefficients equal zero, therefore, $y=1$, and if it were not for this lucky circumstance the values of x and y, in (1) and (2) would still be far away.

We have observed, that all remainders in these operations, must be zero ; therefore, our first remainder must be zero, and $x=y-2y^3$.

Or, $x=(1-2y^2)y$. This value of x put in (2) and we have $(1+2y^2)^3y^3+y^3=0$. Or, $(1-2y^2)^3+1=0$.

Expanding and dividing by 2, produces

$4y^6-6y^4+3y^2-1=0$. The same as before.

But if the object is to solve equations (1) and (2), and find the values of x and y, the common method is incomparably the best.

$$x^2+xy+y^2=1. \qquad (1)$$

Equation (2) is divisible by $(x+y)$, and the other factor is x^2-xy+y^2. Therefore we have

$$x+y=0. \qquad (2)$$

Or, $$x^2-xy+y^2=0. \qquad (3)$$

Subtracting (3) from (1) and we obtain

$2xy=1$, or, $xy=\frac{1}{2}$.

This last added to (1), and subtracted from (3) will produce

$$x^2+2xy+y^2=\tfrac{3}{2},$$

And, $$x^2-2xy+y^2=-\tfrac{1}{2},$$

$$x+y=\pm\sqrt{\tfrac{3}{2}}$$

$$x-y=\pm\sqrt{-\tfrac{1}{2}}$$

$x=\pm\frac{1}{4}(\sqrt{6}+\sqrt{-2})$ $y=\pm\frac{1}{4}(\sqrt{6}-\sqrt{-2})$.

Also, $x=-1$; and $y=1$.

APPENDIX.

DESIGNED TO SHOW THE UTILITY OF ALGEBRA IN PHILOSOPHICAL INVESTIGATIONS.

INEQUALITY,

Expressions of inequality sometimes occur in the mathematics, but we can always make equations of them by adding a symbol to the smaller quantity. Then we can reduce the expressions the same as equations, without danger of confusion.

EXAMPLES.

1. *Find the limit of the value of* x *in the inequation*

$$7x-\frac{23}{3}>\frac{2x}{3}+5.$$

Let h represent the difference between these two expressions,

then, ... $7x-\frac{23}{3}=\frac{2x}{3}+5+h$, is a perfect equation

Whence, ... $21x-23=2x+15+3h$.

$$19x=38+3h, \text{ or } x=2+\tfrac{3}{19}h.$$

That is, x is greater than the number 2.

2. *The double of a number diminished by* 5 *is greater than* 25, *and triple of the number diminished by* 7 *is less than the double increased by* 13. *What numbers will satisfy the conditions?*

Let $x=$ the number, and h and k the difference of the expressions.

Then, ... $2x-5=25+h$. (1)

$3x-7+k=2x+13$. (2)

From . . . . (1) $x=15+\frac{1}{2}h$. From (2) $x=20-k$.

That is, x must be more than 15 and less than 20, consequently, 16, 17, 18 and 19 will each answer the conditions.

3. *The sum of two numbers is* 32, *and if the greater be divided by the less, the quotient will be less than* 5, *but greater than* 2. *What are the numbers?*

Let $x=$ the greater, $y=$ the less, then $x+y=32$. (1)

And . . . . $\frac{x}{y}=5-h$. (2) $\frac{x}{y}=2+k$. (3)

From (2) $x=5y-hy$. This put in (1) gives $6y-hy=32$.

Therefore, $y=\frac{32}{6-h}$. From (3) $x=2y+ky$; this, put in (1), gives $y=\frac{32}{3+k}$.

From these last equations we perceive that 32 divided by less than 6 will give the limit of y in one direction, and divided by more than 3 will give its limit in the opposite direction; that is, y must be more than 5 and less than 10; consequently x must be more than 22 and less than 27.

DIFFERENTIAL METHOD OF SERIES.

A series purely of the arithmetical order has been fully treated, as also those of the geometrical order; but a series in general exist, when each of the terms is derived from one or more of the preceding terms, according to some definite law. (Art. 141)

Therefore, a variety of series may exist, neither arithmetical nor geometrical, and these can be investigated by means of *finite differences;* and the word differential at the head of these remarks, is not identical with the same word applied to the differential calculus. For example,

1, 5, 15, 35, 70, 126,

and so on, is a regular series, and if we wished to find its 10th, 12th, or *n*th terms, it would be too unscientific and tedious to find it through the succession of terms ; we must, therefore, investigate for general principles.

The series is 1, 5, 15, 35, 70, 126, 210,
1st order of diff. 4, 10, 20, 35, 56, 84,
2d " " " 6, 10, 15, 21, 28,
3d " " " 4, 5, 6, 7,
4th " " " 1, 1, 1,
5th " " " 0, 0,

Here it will be observed that the first term of the series is 1. The first term of the first order of differences is 4. The first term of the 2d order of differences is 6. The first term of the 3d order of differences is 4. The first term of the 4th order is 1 ; and lastly, the first term and all the terms of the 5th order are 0.

To obtain general principles, however, we must take general notation as follows: (The symbol a_4 is read a sub. 4, and so on for other like symbols.)

Let . . . . . $a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_n,$ be a series.
$b_1,\ b_2,\ b_3,\ b_4,\ b_n,$ 1st differences
$c_1,\ c_2,\ c_3,\ c_4,$ 2d differences.
$d_1,\ d_2,\ d_3,$ 3d differences.
&c. &c.

By this notation we perceive that d_4 would represent the 4th term of the third order of differences, d being the third letter of the alphabet.

From the foregoing notation, we readily derive the following equations:

$$\begin{array}{lll} a_2-a_1=b_1, & b_2-b_1=c_1, & c_2-c_1=d_1, \\ a_3-a_2=b_2, & b_3-b_2=c_2, & c_3-c_2=d_2, \\ a_4-a_3=b_3, & b_4-b_3=c_3, & c_4-c_3=d_3, \\ a_5-a_4=b_4, & b_5-b_4=c_4, & c_5-c_4=d_4, \\ \text{and so on,} & \text{and so on,} & \text{and so on.} \end{array}$$

By transposition (writing the letters in alphabetical order), we have

$$\begin{array}{lll} a_2=a_1+b_1, & b_2=b_1+c_1, & c_2=c_1+d_1, \\ a_3=a_2+b_2, & b_3=b_2+c_2, & c_3=c_2+d_2, \\ a_4=a_3+b_3, & b_4=b_3+c_3, & c_4=c_3+d_3, \\ a_5=a_4+b_4, & b_5=b_4+c_4, & c_5=c_4+d_4, \\ \text{and so on,} & \text{and so on,} & \text{and so on.} \end{array}$$

In the 2d equation of the first column we will substitute the values of a_2 and b_2. then we shall have $a_3=a_1+2b_1+c_1$. (1)

By the same process we find

$$b_3 = b_1 + 2c_1 + d_1. \qquad (2)$$

In the 3d equation of the first column we find

$$a_4 = a_3 + b_3.$$

Substituting the values of a_3, b_3 from equations (1) and (2) gives us

$$a_4 = a_1 + 3b_1 + 3c_1 + d_1. \qquad (3)$$

By the same process,

$$b_4 = b_1 + 3c_1 + 3d_1 + e_1. \qquad (4)$$

Also by referring back we find

$$a_5 = a_4 + b_4.$$

Substituting the values of a_4 and b_4 from (3) and (4), we find

$$a_5 = a_1 + 4b_1 + 6c_1 + 4d_1 + e_1. \qquad (5)$$

By inspecting equations (1), (3) and (5), we perceive that the coefficients in the 2d members correspond to binomial coefficients, and if we should continue the process to any degree, they would still correspond, *and the letters represent the first terms of the several orders of differences.*

Therefore, in general,

$$a_{n+1} = a + nb + n\,.\,\frac{n-1}{2}c + n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}d + \ \&c.$$

And . . $$b_{n+1} = b + nc + n\,.\,\frac{n-1}{2}d + n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}e + \ \&c.$$

We drop the sub. ones in the second members.

The series terminates with that order of difference which becomes constant; for the succeeding order becomes zero.

EXAMPLES.

1. Required the 12th term of the series.

1, 5, 15, 35, 70, 126,
4, 10, 20, 35, 56,
6, 10, 15, 21,
4, 5, 6,
1, 1.

Here $a=1$, $b=4$, $c=6$, $d=4$, $e=1$, $f=0$, $n=11$.

Whence, $a_{12} = 1 + 11\,.\,4 + 11\,.\,\frac{10}{2}\,.\,6 + 11\,.\,\frac{10}{2}\,.\,\frac{9}{3}\,.\,4 + 11\,.\,\frac{10}{2}\;\frac{9}{3}\,.\,\frac{8}{4}.$

Or, . . . $a_{12} = 1 + 44 + 330 + 660 + 330 = 1365$, *Ans*

2. Required the 15th term of the series

1, 4, 10, 20, 35, &c. *Ans.* 670

3. Required the 20th term of the series

6, 10, 15, 21, &c. *Ans.* 253.

4. Required the nth term of the series

1, 3, 6, 10, &c. *Ans.* $\frac{n(n+1)}{2}$.

5. Required the nth term of the series

2, 6, 12, 20, 30, &c. *Ans.* $n^2 + n$.

N. B. To solve the 4th and 5th examples, write $(n-1)$ for n in the formula.

6. Find the 20th term of the series 1, 8, 27, 64, 125, &c. *Ans.* 8000.

7. Required the 50th term of the series

1, 3, 6, 10, 15, &c. *Ans.* 1275.

Compare examples 4 and 7.

Our investigation thus far has been limited to finding a particular term of a series.

We now propose to find the sum of any proposed number of consecutive terms.

Let . . . a_1, a_2, a_3, a_4, a_5, . . . a_n, &c., represent a series.

With the terms of this series form a new series commencing with 0, for its first term; adding the succeeding term of the proposed series to the preceding term of the new series, thus,

$$0, \quad (0+a_1), \quad (0+a_1+a_2), \quad (0+a_1+a_2+a_3), \quad \text{and so on.}$$

Here it is obvious that the $(n+1)^{\text{th}}$ term of this new series is the sum of n terms of the proposed series. Therefore, we will find the $(n+1)^{\text{th}}$ term of the new series by the formula, and that will be n terms of the proposed series as required.

EXAMPLES.

1. Find the sum of 10 terms of the series 3, 5, 7, 9, 11, &c.

The new series to be formed is

$$0, \quad (0+3), \quad (0+3+5), \quad (0+3+5+7), \quad (0+3+5+7+9), \quad \&c$$

That is, ... 0, 3, 8, 15, 24, &c.

We must now find the 11th term of this series, which will be the sum of 10 terms of the proposed series as required. *Ans.* 120.

2. Find the sum of 20 terms; also the sum of n terms of the series

1, 3, 6, 10, 15, &c.

Ans. 20 terms $=1540$, n terms $=\dfrac{n(n+1)(n+2)}{6}$.

3. Find n terms of the series 2, 6, 12, 20, 30, &c.

Ans. $\dfrac{n(n+1)(n+2)}{3}$

4. Find the sum of n terms of the series of cubes

1^3, 2^3, 3^3, 4^3, &c. *Ans.* $\left(\dfrac{n^2+n}{2}\right)^2$.

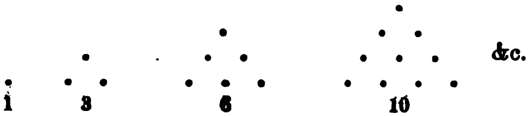
5. Find the sum of 10 terms, also the sum of n terms of the series

1, 4, 9, 16, 25, 36, &c.

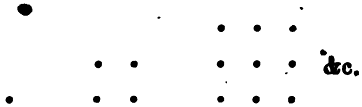
N. B. The new series is 0, 1, 5, 14, 30, 55, 91, &c

Ans. Sum of 10 terms $=385$. Sum of n terms $=\dfrac{(2n^2+3n+1)n}{6}$.

The series of numbers in example 2 are called triangular numbers, because they may be represented by points forming equilateral triangles, thus:



In like manner the square numbers in example 5 can be represented by points, thus:



In forts and armories shot and shells are piled on triangular, square, or oblong bases.

When the base is a triangle, and the pile complete, the number of balls in the pile is expressed by the answer to example 2, in which n represents the number of layers in the pile.

When the base is a square, and the pile complete, the number of shot in the pile is expressed by the answer to example 5, in which n represents the number of layers in the pile.

When the pile is an oblong and complete, the upper layer is a single row. The next larger has two rows and one more ball in each row. The third layer consists of three rows, and each row one more ball than the preceding.

Now if the top layer consists of			$m+1$	balls,
The 2d must consist of . . .	$2(m+2)$	or	$2m+4$	"
The 3d must consist of . . .	$3(m+3)$	or	$3m+9$	"
The 4th must consist of . . .	$4(m+4)$	or	$4m+16$	"
The 5th must consist of . . .	$5(m+5)$	or	$5m+25$	" &c.

The number of balls in the whole pile must then consist of the sum of the series $(m+1)$, $(2m+4)$, $(3m+9)$, &c., to n terms n representing the number of layers in any pile.

We therefore require the sum of this series to n *terms.*

The new series to be formed is

	0,	$(0+m+1)$,	$(0+3m+5)$,	$(0+6m+14)$,	&c.
1st diff.	$(m+1)$,	$(2m+4)$,	$(3m+9)$,	$(4m+16)$,	&c.
2d diff.		$(m+3)$,	$(m+5)$,	$(m+7)$.	
3d diff.			2	2	
4th diff.			0		

Now by the formula, the $(n+1)^{\text{th}}$ term of this series is

$$a_{n+1}=0+n(m+1)+n\,.\,\tfrac{n-1}{2}(m+3)+n\,.\,\tfrac{n-1}{2}\,.\,\tfrac{n-2}{3}\times 2.$$

The right hand member of this equation is the general formula for an oblong pile of shot or shells, in which $(m+1)$ represents the number in the top row, and n the number of layers.

We have previously found that $\dfrac{n(n+1)(n+2)}{6}$ represents a triangular pyramidical pile and, $\dfrac{n(2n^2+3n+1)}{6}$ represents a square pyramidical pile

EXAMPLES.

1. How many shot are in a triangular pyramidical pile consisting of 18 layers? *Ans.* 1140.

2. How many shot or shells in a square pile consisting of 15 layers? *Ans.* 1240.

3. The top row of an oblong pile of shot or shells consists of 31 balls, and the number of layers is 30. How many balls are there in the whole pile? *Ans.* 23405.

4. There is an oblong pile of balls consisting of 20 layers, and 6440 balls ; how many balls are in its base ? *Ans*. 740.

5. A square pile of shells consists of 12 layers, the upper layer has 8 shells on a side ; how many shells are in the pile ? *Ans*. 2330.

6. The number of balls in a triangular pile is to the number in a square pile (having the same number of layers) as 5 to 9 ; required the number in each pile. *Ans*. 455 and 819.

Our object is now to express the several orders of differences by the terms of the series,

$$\begin{array}{ll} & a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad \&c. \\ \text{1st order,} & (a_2-a_1),\ (a_3-a_2),\ (a_4-a_3),\ (a_5-a_4)\ \&c. \\ \text{2d order,} & (a_3-2a_2+a_1),\ (a_4-2a_3+a_2),\ (a_5-2a_4+a_3). \\ \text{3d order,} & (a_4-3a_3+3a_2-a_1),\ (a^5-3a_4+3a_3-a\ ) \\ \text{4th order,} & (a_5-4a_4+6a_3-4a\ +a_1). \end{array}$$

Here it can be observed that the coefficients correspond to those of the powers of a binomial.

Also observe that to compose the *first* term of the 2d order of differences, we must use the first three terms of the series. To compose the first term of the 3d order, we must use the first four terms of the series. To compose the first term of the 4th order of differences, we must use the first five terms of the series: and in general, to compose the first term of the n^{th} order of differences, we must use the first $(n+1)$ terms of the series. Observing these facts, the first term of the 5th order of differences of the preceding general series, must be expressed thus:

$$a_6-5a_5+10a_4-10a_3+5a_2-a_1.$$

If the first term of any order vanishes, or becomes very small, the expression for it may be put equal to zero, and any term of the series comprised in it, can be found, provided the other terms are given.

For example, suppose the 4th order of differences in a series becomes 0, then

$$a_5-4a_4+6a_3-4a_2+a_1=0.$$

Now suppose the 3d term of the series lost or unknown, the others being given, it is found thus:

$$a_3=\frac{4a_4+4a_2-a_5-a_1}{6}.$$

The series 1, 8, a_3, 64, 125, has lost its third term, its 4th order of difference vanishes; what is that lost term?

$$a_3=\frac{4\ .\ 64+4\ .\ 8-125-1}{6}=\frac{288-126}{6}=27 \ \textit{Ans.}$$

One of the most important applications of this calculus of finite differences is, that of inserting between the terms of a series some new term or terms subject to the same law. This is called

INTERPOLATION.

We have already seen that the general equation is

$$a_{n+1}=a+nb+n\ .\ \frac{n-1}{2}c+n\ .\ \frac{n-1}{2}\ .\ \frac{n-2}{3}d, \ \&c.$$

This series will apply to any term beyond a. If we required the 3d term beyond a, we must put $n=3$. If the 4th beyond a, we put $n=4$, and so on.

If part of a term beyond a, we put $n=$ to that fractional part.

EXAMPLE.

Given the logarithms of 102, 103, 104, and 105, to find the logarithm of 103.6, which is the 1.6th term beyond a.

NO.	LOGARI'MS.	1ST DIFF.	2D DIFF.	3D DIFF.
102	2. 0086002			
103	2. 0128372	.0042370		
104	2. 0170333	.0041961	—.0000409	
105	2. 0211893	.0041560	—.0000401	—.0000008

$a=2.0086002$, $b=.0042370$, $c=-.0000409$, $d=.0000008$, $n=1.6$

$$\begin{array}{ll}
a \ldots\ldots\ldots\ldots\ldots\ldots\ldots & 2.\ 0086002 \\
nb \ldots\ldots\ldots\ldots\ldots\ldots & +\ 0.\ 0067792 \\
n\ .\ \dfrac{n-1}{2}c \ldots\ldots\ldots & -\ 0.\ 0000196 \\
n\ .\ \dfrac{n-1}{2}\ .\ \dfrac{n-2}{3}d \ldots\ldots & +\ 0.\ 00000005 \\
\hline
\text{log. } 103.6 & 2\ \ 01535985
\end{array}$$

In most cases the 3d difference may be omitted: in this case it only influenced the 8th decimal place. If we were to require the logarithm of 102.3, then $n=0.3$. If 104.7, then $n=2.7$, and so on.

Interpolation is much used in astronomy, as the following example will show:

At noon (Greenwich time), May 10th, 1846, the moon's declination was 3° 27′ 17.5″ north; at midnight following it was 5° 46′ 38″ north; on the 11th, at noon, 7° 38′ 57″ north, and the following midnight 10° 2′ 31.7″ north. What was the declination on the 10th of May at 3h P. M?

Here $a=3°\ 27'\ 17.5''$, $b=2°\ 19'\ 20.5''$, $c=-7'\ 1.5''$, $d=-1'\ 43.2''$, and the interval of 12 hours must be considered as the unit of time: 3 hours is, therefore, ¼ of this unit. Hence, $n=0.25$, and the required result must be the sum of the following series:

$$(3°27'\,17.5'')+\tfrac{1}{4}(2°19'\,20.5'')+\tfrac{1}{4}\cdot\frac{\frac{1}{4}-1}{2}\left(\overline{7'\,15''}\right)+\tfrac{1}{4}\cdot\frac{\frac{1}{4}-1}{2}\cdot\frac{\frac{1}{4}-2}{3}\left(\overline{1'\,43.2''}\right)$$

$$\begin{array}{lll}
\text{The 1st} & \text{term } a = & 3°\,27'\ 17.5'' \\
\text{2d} & \text{term } nb & 34'\ 50.1'' \\
\text{3d} & \text{term} & 39.5''. \\
\text{4th} & \text{term} & -\ 5.6''. \\
\hline
 & \text{Sum} & 4°\ 2'\ 41.5''.\ \ \textit{Ans.}
\end{array}$$

To show the utility and application of this formula, we give the following

EXAMPLES.

Given the logarithmic sine of 5°, 5° 12′, 5° 24′, 5° 36′, 5° 48′ *and* 6°, *to find the sine of any or all intermediate arcs.*

	a.	+*b.* 1st diff.	—*c.* 2d diff.	+*d.* 3d diff.	—*e.* 4th diff.
Sine 5° 0′=8.	9402360				
Sine 5° 12′=8.	9572843	.0169883			
Sine 5° 24′=8	9736280	.0163437	6446		
Sine 5° 36′=8	9893737	.0157457	5980	466	
Sine 5° 48′=9	0045634	.0151897	5560	420	46
Sine 6° 0′=9	0192346	.0146712	5185	375	45

In applying the formula to this example, 12′ must be taken as the *unit* of space between the terms. Then if we require the sine of 5° 3′, we must put $n=\frac{3}{12}=\frac{1}{4}$, and the formula gives

$$\text{Sine } 5^\circ\, 3' = 8.9402960+\tfrac{1}{4}(.0169883)+\tfrac{1}{4}\cdot\frac{\frac{1}{4}-1}{2}\left(\overline{6446}\right)+\tfrac{1}{4}\cdot\frac{\frac{1}{4}-1}{2}\cdot\frac{\frac{1}{4}-2}{3}\times 466=8.9402960+.0042471+604.3+25.6=8.9446061.$$

$$\text{Sine } 5^\circ\, 4' = 8.9402960+\tfrac{1}{3}(.0169883)+\tfrac{1}{3}\cdot\frac{\frac{1}{3}-1}{2}\left(\overline{6446}\right)+\tfrac{1}{3}\cdot\frac{\frac{1}{3}-1}{2}\cdot\frac{\frac{1}{3}-2}{3}\times 456=8.9402960+.0056628+716+29=8.9460333.$$

For the *sine* 5° 13′, we put $n=\frac{1}{12}$ and

$$\text{Sn. } 5^\circ\, 13' = 8.9572843+\tfrac{1}{12}(.0163437)+\tfrac{1}{12}\cdot\frac{\frac{1}{12}-1}{2}\left(\overline{5980}\right)+\tfrac{1}{12}\cdot\frac{\frac{1}{12}-1}{2}\times\frac{\frac{1}{12}-2}{3}(420)=8.9572843+.0013619+229+13=8.9586704.$$

To find the log. sine of 5° 14′ 13″, we could put $n=\dfrac{2\frac{31}{60}}{12}=\dfrac{133}{720}$.

The work of finding sines, cosines, and tangents, &c., is already done; and, therefore, in respect to this application, the formula for interpolation has comparatively lost its importance; but with the practical astronomer, this formula is of the greatest importance, and must forever be in use.

We give the following

EXAMPLE.

For the English Nautical Almanac the moon's declination is required for every hour of Greenwich time; but to compute it directly from the lunar tables and trigonometry, would present an apalling amount of labor, and to surmount this difficulty, the declination is computed from the lunar tables for each noon and midnight of Greenwich time; and the declination inserted for the intermediate hours by interpolation.

Opening the English Nautical Almanac for 1854, at page 91, I find the moon's declination as follows:

	☽ dec. *N.*	$+b$	$-c$	$-d$	$+e$
May 23, mean moon,	6° 13′ 14.4″				
May 23, midnight,	8° 55′ 6.4″	2° 41′ 52.0″			
May 24, noon,	11° 30′ 26.2″	2° 35′ 19.8″	6′ 32.2″		
May 24, midnight,	13° 57′ 44.4″	2° 27′ 16.2″	8′ 3.6″	1′ 31.4″	
May 25, noon,	16° 15′ 33.6″	2° 17′ 49.2″	9′ 27.0″	1′ 23.4′	8″

From the above data we require the moon's declination at the commencement of every hour between noon and noon of May 23d and 24th.

For the first hour, or 1, afternoon, we must put $n=\frac{1}{12}$.

$$\text{The ☽'s dec. at 1 P. M.} = 6^\circ\, 13'\, 14.4''+\tfrac{1}{12}(2^\circ\, 41'\, 52.0'')+\tfrac{1}{12}\cdot\frac{\frac{1}{12}-1}{2}\times\left(\overline{6'\, 32.2''}\right)+\tfrac{1}{12}\left(\frac{\frac{1}{12}-1}{2}\right)\left(\frac{\frac{1}{12}-2}{3}\right)\left(\overline{1'\, 31.4''}\right)=6^\circ\, 13'\, 14.4''+13'\, 29.3''+14.9''-2.2''=6^\circ\, 26'\, 56.4''\ \text{North.}$$

The following are the results taken from the Nautical Almanac which serve as answers to twenty-two examples in the application of this formula

Hours.	☽ Dec.	Hours.	☽ Dec.
1	6° 26′ 56.5″	13	9° 8′ 19.3″
2	6 40 36.5	14	9 21 29.4
3	6 54 14.3	15	9 34 36.6
4	7 7 49.8	16	9 47 41.0
5	7 21 23.0	17	10 0 42.5
6	7 34 53.9	18	10 13 40.9
7	7 48 22.3	19	10 26 36.4
8	8 1 48.3	20	10 39 28.7
9	8 15 11.7	21	10 52 17.9
10	8 28 32.6	22	11 5 3.9
11	8 41 50.8	23	11 17 46.7
12	8 55 6.4	24	11 30 26.2

SPECIFIC GRAVITY.

Gravity is weight. Specific gravity is the specified weight of one body, compared with the specified weight of another body (of the same bulk), taken as a standard.

Pure water, at the common temperature of 60° Fahrenheit, is the standard for solids and liquids; common air is the standard for gases.

Water will buoy up its own weight. If a body is lighter than water, it will float; if heavier than water, it will sink in water.

If a body weighs 16 pounds, in air, and when suspended in water weighs only 14 pounds, it is clear that its bulk of water weighs 2 pounds; and the body is 8 times heavier than water; therefore the specific gravity of this body is 8, water being 1.

If the specific gravity of a body is n, it means that it is n times heavier than its bulk of water. Therefore —

If we divide the weight of any body by its specific gravity, the quotient will be, the weight of its bulk of water.

On this fact alone we may resolve all questions pertaining to specific gravity

EXAMPLES.

1. Two bodies, whose weights were A and B, and specific gravities a and b, were put together in such proportions as to make the specific gravity of the compound mass c. What proportions of A and B were taken?

A quantity of water, equal in bulk to A, must weigh $\frac{A}{a}$

A quantity " " " " B, " " $\frac{B}{a}$

A quantity of water, equal in bulk to $(A+B)$, will weigh $\frac{A+B}{c}$

Therefore, $\frac{A}{a}+\frac{B}{b}=\frac{A+B}{c}$; Or, $bcA+acB=abA+abB$;

Or, $b(c-a)A=a(b-c)B$.

Hence the quantities of each must be reciprocal to these coefficients; or if we take one, or unity of B, we must take $\frac{a}{b}\left(\frac{b-c}{c-a}\right)$ units of A.

2. Hiero, king of Sicily, sent gold to his jeweler to make him a crown; he afterwards suspected that the jeweler had retained a portion of the gold, and substituted the same weight of silver, and he employed Archimedes to ascertain the fact, who, after due reflection, hit upon the expedient of specific gravity

He found, by accurately weighing the bodies both in and out of water, that the specific gravity of gold was 19, of silver 10.5, and of the crown 16.5. From these data he found what portion of the king's gold was purloined. Repeat the process.

The preceding problem is the abstract of this, in which A may represent the weight of the gold in the crown, B the weight of the silver, and $(A+B)$ the weight of the crown; $a=19$, $b=10\frac{1}{2}$, $c=16\frac{1}{2}$.

Then if we take $B=1$, one pound, one ounce, or any unity of weight, of silver, the comparative weight of the gold will be expressed by $\frac{a}{b}\left(\frac{b-c}{c-a}\right)$.

That is, for every ounce of silver in the crown, there were $4\frac{2}{35}$ ounces of gold. If clearer to the pupil, he may resolve this problem as an original one, without substituting from the abstract problem.

3. I wish to obtain the specific gravity of a piece of wood that weighs 10 pounds; and as it will float on water, I attach 21 pounds of copper to it, of a specific gravity of 9. The whole mass, 31 pounds, when weighed in water, weighs only 4 pounds; hence 27 pounds of the 31 were buoyed up by the water; or we may say, the same bulk of water weighed 27 pounds. Required the specific gravity of the wood.

Let s represent the specific gravity of the wood.

Then $\frac{10}{s}=$ the weight of the same bulk of water.

And $\frac{21}{9}=$ the weight of water of the same bulk as the copper.

Hence, ... $\frac{10}{s}+\frac{7}{3}=27$, Or, $s=\frac{30}{74}=.405$, nearly.

4. Granite rock has a specific gravity of 3. A piece that weighs 30 ounces, being weighed in a fluid, was found to weigh only 21.5 ounces. What was the specific gravity of that fluid?

The weight of the fluid, of the same bulk as the piece of granite, was evidently 8.5 ounces. Let s represent its specific gravity.

Then $\frac{8.5}{s}=$ the weight of the same bulk of water; also $\frac{30}{3}=10=$ the weight of the same bulk of water.

Hence $\frac{8.5}{s}=10$, Or, $s=.85$ *Ans.*, which indicates impure alcohol

5. The specific gravity of pure alcohol is .797; a quantity is offered of the specific gravity of .85, what proportion of water does it contain?

Let $A=$ the pure alcohol, and $W=$ the water.

Then $\frac{A}{.797}+W=\frac{A+W}{.85}$.

Ans. The resolution of this equation shows 1 portion of water to 2.255+ portions of alcohol.

6. There is a block of marble, in the walls of Balbeck, 63 feet long, 12 wide, and 12 high. What is the weight of it in tons, the specific gravity of marble being 2.7 and a cubic foot of water $62\frac{1}{2}$ pounds, *Ans.* $683\frac{4}{10}$ tons.

7. The specific gravity of dry oak is 0.925; what, then, is the weight of a dry oak log, 20 feet in length, 3 feet broad, and $2\frac{1}{2}$ feet deep? *Ans.* $8671\frac{7}{8}$ lbs.

We may now change the subject, to make a little examination into *maxima* and *minima*. For this purpose, let us examine Problem 2 (Art. 114).

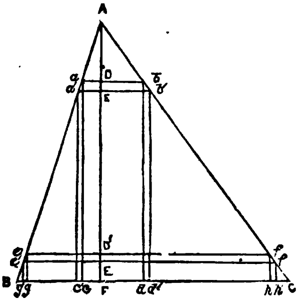
1. Divide 20 into two such parts that their product shall be 140. It may be impossible to fulfil this requisition, therefore we will change it as follows:

Divide 20 into two such parts, that their product will be the *greatest possible*.

Let $x+y=$ one part, and $x-y=$ the other part.

Then $2x=20$, and $x=10$, and the product, x^2-y^2, is evidently the greatest possible when $y=0$. Hence the two parts are equal, and the greatest product is 100, or the square of one half the given number.

2. Given the base, m, and the perpendicular, n, of a plane triangle, to find the greatest possible rectangle that can be inscribed in the triangle.*



Let ABC be the triangle, $BC=m$, $AF=n$, $AD=x$, and $AE=x'$, De be a *very* small distance, so that x' is but insensibly greater than x.

As D, comparatively, is not far from the vertex, it is visible, that the rectangle $a'b'c'd'$ is *greater* than the rectangle $abcd$.

If we conceive the upper side of the rectangle to pass through D', in place of D, and we represent AD' by x, and Ae by x', it is visible that the rectangle $e'f'g'h'$ is *less* than the rectangle $efgh$.

If we subtract the rectangle $abcd$ from the rectangle $a'b'c'd'$, we shall have a *positive* remainder.

If we subtract the rectangle $efgh$ from the rectangle $e'f'g'h'$, we shall have a *negative* remainder.

---

* We do not introduce this problem to show its solution; it belongs to the calculus, and, in its place, is extremely simple; we introduce it to show a principle of reasoning extensively used in the higher mathematics, and, perchance our illustration may aid a pupil in his progress in the calculus.

The rectangle $a\,b\,c\,d$ cannot be the greatest possible, so long as we can have a positive remainder by subtracting it from the next consecutive rectangle immediately below.

After we pass the point on the line AF where the greatest possible rectangle comes in, the next consecutive rectangle immediately below, will become less; and by subtracting the upper from it, the difference will be negative.

Hence, when $a\,b\,c\,d$ becomes the greatest possible rectangle, the difference between it and its next consecutive rectangle can be neither *plus* nor *minus* but must be *zero*.

Therefore, it is manifest, that if we obtain two algebraical expressions for the two rectangles $a\,b\,c\,d$ and $a'b'c'd'$, and put their difference equal to 0, a resolution of the equation will point out the position and magnitude of the *maximum* rectangle required.

Put the line $ab=y$, and $a'b'=y'$. As $AD=x$ and $AE=x'$, $DF=n-x$ and $EF=n-x'$. The rectangle $a\,b\,c\,d=y(n-x)$, and $a'b'c'd'=y'(n-x')$.

From the consideration just given, the maximum must give

$$y'(n-x')-y(n-x)=0.$$

By proportional triangles, we have $x : y :: n : m$ or, $y=\frac{mx}{n}$.

By a like proportion, we have . . . . $y'=\frac{m}{n}x'$.

Put these values of y and y' in the above equation, and, dividing by $\frac{m}{n}$, we have . . . . $x'(n-x')=x(n-x)$;

Or, ... $x^2-x'^2=n(x-x')$.

By division, ... $x+x'=n$

As x' is but insensibly greater than x, $2x=n$; which shows that AD is one half AF, and the greatest rectangle must have just half the altitude of the triangle.

3. *Required the greatest possible cylinder that can be cut from a right cone.*

Conceive the triangle (of Prob. 2.) to show the vertical plane cut through the vertex of the cone, and $ab=y$ the diameter of the required cylinder. Then, the end of the cylinder is $.7854y^2$, and its solidity is $.7854y^2(n-x)$. The next consecutive cylinder is $.7854y'2(n-x')$. Hence $y'^2(n-x')=y^2(n-x)$.

By similar triangles $x : y :: n : m$, Or, $y^2=\frac{m^2x^2}{n^2}$ and $y'^2=\frac{m^2}{n^2}x'^2$.

Hence, $x'^2(n-x')=x^2(n-x)$, Or, $x^3-x'^3=n(x^2-x'^2)$;

Divide both members by $(x-x')$, and $x^2+xx'+x'^2=n(x+x')$.

As $x=x'$ infinitely near, $3x^2=2nx$, or, $x=\frac{2}{3}n$; which shows that the altitude of the maximum cylinder is $\frac{1}{3}$ the altitude of the cone.

In this way all problems pertaining to maxima and minima can be resolved; but the notation and language of the calculus, *in all its bearings*, is preferable to this. We had but a single object in view—that of showing the power of algebra.

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